Lecture 4

Important mathematical properties of a Lorentz transformation. Causal classification of events [T-W 6.1, 6.2, 6.3 (1.7)]
A Lorentz transformation has several invariants. They are so geometrical that they can be inferred immediately from a space-time diagram.

A Lorentz transformation preserves parallelism and perpendicularity. It also preserves hyperbolas of revolution \( t^2 - x^2 - y^2 = \text{const} \) and their degenerate limits, the light cones \( t^2 - x^2 - y^2 - z^2 = 0 \).

1. Being the spacetime analogue of Euclidean rotations, the Lorentz rotations can be seen to leave invariant
   (1) Hyperbolas, including their degenerate limits.
   (2) Parallelness between lines.
   (3) Perpendicularity between lines.
Hyperbola as a geometrical invariant of the Lorentz transformation

\[ t = \tau \cos \theta + \tau \sin \theta \]

\[ x = \tau \sin \theta + \tau \cos \theta \]

A calculation yields

\[ (t-t_0)^2 - (x-x_0)^2 = (\tau \cos \theta + \tau \sin \theta - t_0)^2 \]

\[ - (\tau \cos \theta + \tau \sin \theta - x_0)^2 \]

\[ = \tau^2 - x^2 + t_0^2 - x_0^2 \]

\[ - 2(\tau t_0 - x_0 \tau) \cos \theta \]

\[ - 2(\tau t_0 - x_0 \tau) \sin \theta \]

\[ = \tau^2 - x^2 + t_0^2 - x_0^2 \]

\[ - 2(\tau t_0 - x_0 \tau) \cos \theta \]

\[ + 2(\tau t_0 - x_0 \tau) \sin \theta \]

\[ = (\tau \cos \theta - x_0 \sin \theta - \tau)^2 \]

\[ - (\tau \cos \theta - x_0 \sin \theta - x_0)^2 \]

Thus

\[ (t-t_0)^2 - (x-x_0)^2 = (t - t_0)^2 - (x - x_0)^2 \]

where we set

\[ \tau_0 = t_0 \cos \theta - x_0 \sin \theta \]

\[ \tau_0 = x_0 \cos \theta - t_0 \sin \theta \]

Note the minus sign as compared with above.

Thus, hyperbolas and their degenerate limits get mapped into hyperbolas and their degenerate limits in a Lorentz transformation.

The Lorentz invariant hyperbolae ("circles" in spacetime) can be used to compare distances (i.e., projections along some axis) in one frame with those in another frame. In particular, one can compare meter rods in two respective frames; one can also compare two clocks in two respective frames. For example, time dilation emerges as a consequence of the relativity of simultaneity.
4.5

Causal Structure of Spacetime.

There is one outstanding feature of spacetime which Euclidean space does not have, namely a so-called causal structure.

This structure is a classification of pairs of events which is the direct consequence of the sign indefiniteness of the invariant interval.

In fact the sign of this interval determines whether two events have a time, space, or lightlike relation.

Thus the following equally valid claims can be made.

Rocket observer: Lab clock reads \( t = 1 \) after
his own clock reads \( E = 1 \), i.e., lab clock is slow.
Lab observer: Rocket clock reads \( E = 1 \) after
his own clock reads \( t = 1 \), i.e., rocket clock is slow.

Typical event \( t^2 - x^2 - y^2 - z^2 = 1 \) interval \( t \) between

Typical event \( t^2 - x^2 - y^2 - z^2 = 1 \) interval \( t \) between

Event plane: Future lightcone of \( P \) A
\( > 0 \) timelike \( t > 0 \)
E
\( > 0 \) \( u \)
\( t < 0 \)
B
\( = 0 \) lightlike \( t > 0 \)
D
\( = 0 \) \( v \)
\( t < 0 \)
C
\( < 0 \) spacelike \( t > 0 \)

Upradable from \( E \)
D
\
\( t < 0 \) 
\
\( t > 0 \)