

Lecture 7

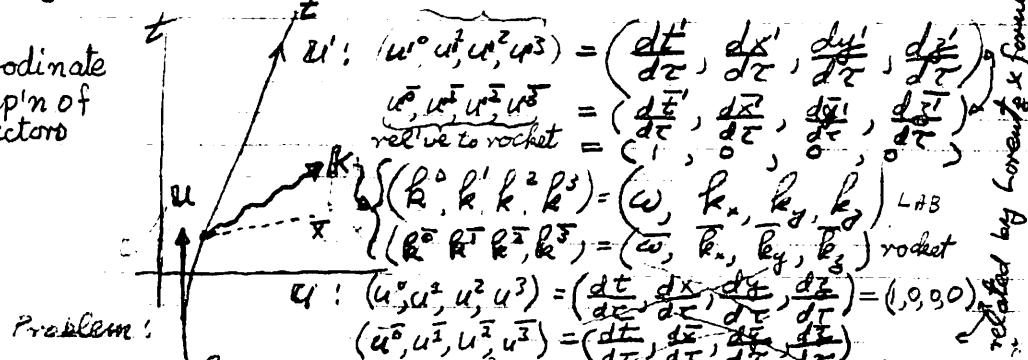
1. Significance and utility of the projection invariant.
2. How and why Einstein formed the concept of "instantaneous Lorentz frames"
[MTW Ch. 6]

7.1 6.

Invariants are important because they express what is physically and geometrically important. They also make computations easy because they can be evaluated in the most convenient Lorentz frame. As an example, consider the following problem:

In the lab frame the worldline of a rocket is seen to have four velocity components $\{u^{\mu}\} = \left\{ \frac{dx^{\mu}}{d\tau} \right\}$. Also a wave with four propagation vector components $\{k^{\mu}\} = \{w, k_x, k_y, k_z\}$ is emitted by the rocket.

relative to LAB

coordinate
rep'n of
vectors

Find the emission frequency relative to the rocket frame, whose time axis

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points along u .

This is an archetypical example. It asks for an invariant which is physically and geometrically significant. This claim is true because

Lemma

$$\text{def } K \equiv -u^0 k^0 + u^1 k^1 + u^2 k^2 + u^3 k^3 = -u^0 k^0 + u^1 k^1 + u^2 k^2 + u^3 k^3$$

is a Lorentz invariant, i.e. independent of the inertial frame relative to the components are given. The primes over the indices indicates components relative to the "primed" frame.

Proof:

That $u \cdot K$ is an invariant follows from the fact that each of the terms on the right hand side of

$$u \cdot K = (u^0 K^0 + u^1 K^1 + u^2 K^2 + u^3 K^3) - u^0 u^0 - K^0 K^0$$

is an invariant.

Being an invariant, let us evaluate it in the cooving rocket frame where

$$(u^0, u^1, u^2, u^3) = (0, 0, 0, 0),$$

i.e. where the spatial origin has zero spatial velocity.

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Such an invariant is important because it provides another way (besides the Lorentz transformation method) for relating the observations in different frames (i.e., "from different perspectives") of the same phenomenon. The method of invariants presupposes Lorentz transformations but it does not make explicit use of them.

Example (Planewave frequency relative to different inertial frames)

GIVEN: (1) A plane wave whose propagation four-vector has components

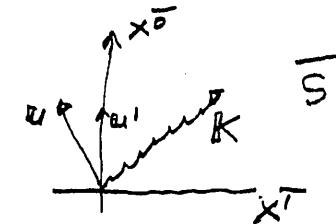
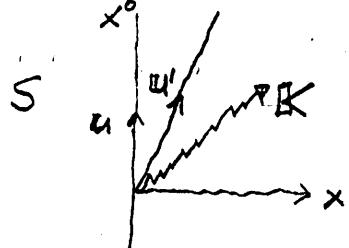
$$K = \{k^0, k^1, k^2, k^3\}$$

with frequency $\omega = k^0$, all relative to the LAB frame S,

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(2). An observer in a ROCKET frame S'. His four-velocity components relative to the LAB frame are

$$u^i = \{u^{i0}, u^{i1}, u^{i2}, u^{i3}\}$$



FIND: Relative to S' the frequency $\bar{\omega}$ of the given wave.

Solution:

a) Relative to S' one has

$$u^i = \{u^{i0}, u^{i1}, u^{i2}, u^{i3}\} = \{1, 0, 0, 0\} \left(\frac{dx^i}{dx} \right)$$

$$K = \{k^0, k^1, k^2, k^3\}$$

$$\omega \cdot K = -u^{i0} k^0 + 0 + 0 + 0 = -k^0 = -\bar{\omega} \quad (= ?)$$

$$= \underbrace{\left(\eta_{\mu\nu} u^{\mu} \bar{k}^{\nu} \right)}_{\text{where } [\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \quad (= ?)$$

b) Relative to S one has

$$\begin{aligned} \mathbf{u}' &= \{u'^0, u'^1, u'^2, u'^3\} \left(= \left\{ \frac{dx'^\mu}{d\tau} \right\} \right) \\ \mathbf{k} &= \{k^0, k^1, k^2, k^3\} \end{aligned} \quad \left. \begin{array}{l} \text{Given,} \\ \text{i.e. known} \end{array} \right\}$$

$$\omega' \cdot \mathbf{k} = -u'^0 k^0 + u'^1 k^1 + u'^2 k^2 + u'^3 k^3$$

c) Invariance of $\omega' \cdot \mathbf{k}$:

$$\begin{aligned} \bar{\omega} &= -\mathbf{u}' \cdot \mathbf{k} = u'^0 k^0 + u'^1 k^1 + u'^2 k^2 + u'^3 k^3 \\ &\quad \text{as calculated relative} \\ &\quad \text{to S.} \\ &= \text{Frequency of wave in} \\ &\quad \text{ROCKET frame, but} \\ &\quad \text{calculated relative to} \\ &\quad \text{the LAB frame.} \end{aligned}$$

d) Conclusion:

Geometrically one has

$$-\bar{\omega} = \mathbf{u}' \cdot \mathbf{k} = \text{projection of } \mathbf{k} \text{ along} \\ \text{unit vector } \mathbf{u}'$$

$$-\omega = \mathbf{u} \cdot \mathbf{k} = \text{projection of } \mathbf{k} \text{ along} \\ \text{unit vector } \mathbf{u}.$$

This projection technique is a shortcut which circumvents the direct use of a Lorentz transformation.

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7.6

Accelerated Frame via
Inertial ("Free Float") Frames.

I. By examining the relation between inertial frames, an accelerated frame and a frame with a static gravitational field, we shall illustrate a type of reasoning which has opened (in this case on the part of Einstein in 1907) new vistas in physics and mathematics.

Other examples of such reasoning are

(i) Galileo's examination of vertical and horizontal motion

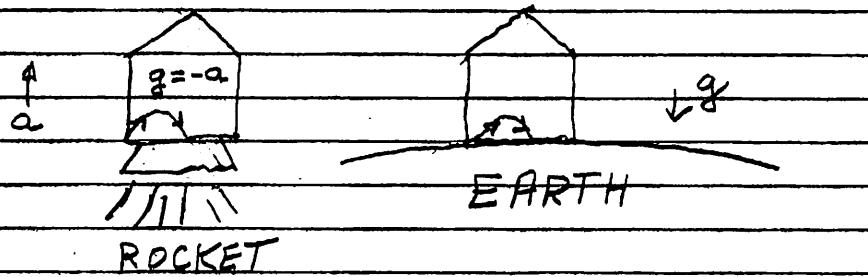
(ii) Newton's analysis of circular motion, which gave rise among others to vector analysis

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(iii) Euler's analysis of the motion of fluids and his analysis of a vibrating membrane

(iv) ...

II. The context of Einstein's 1907 analysis of an accelerated frame was is the observation that



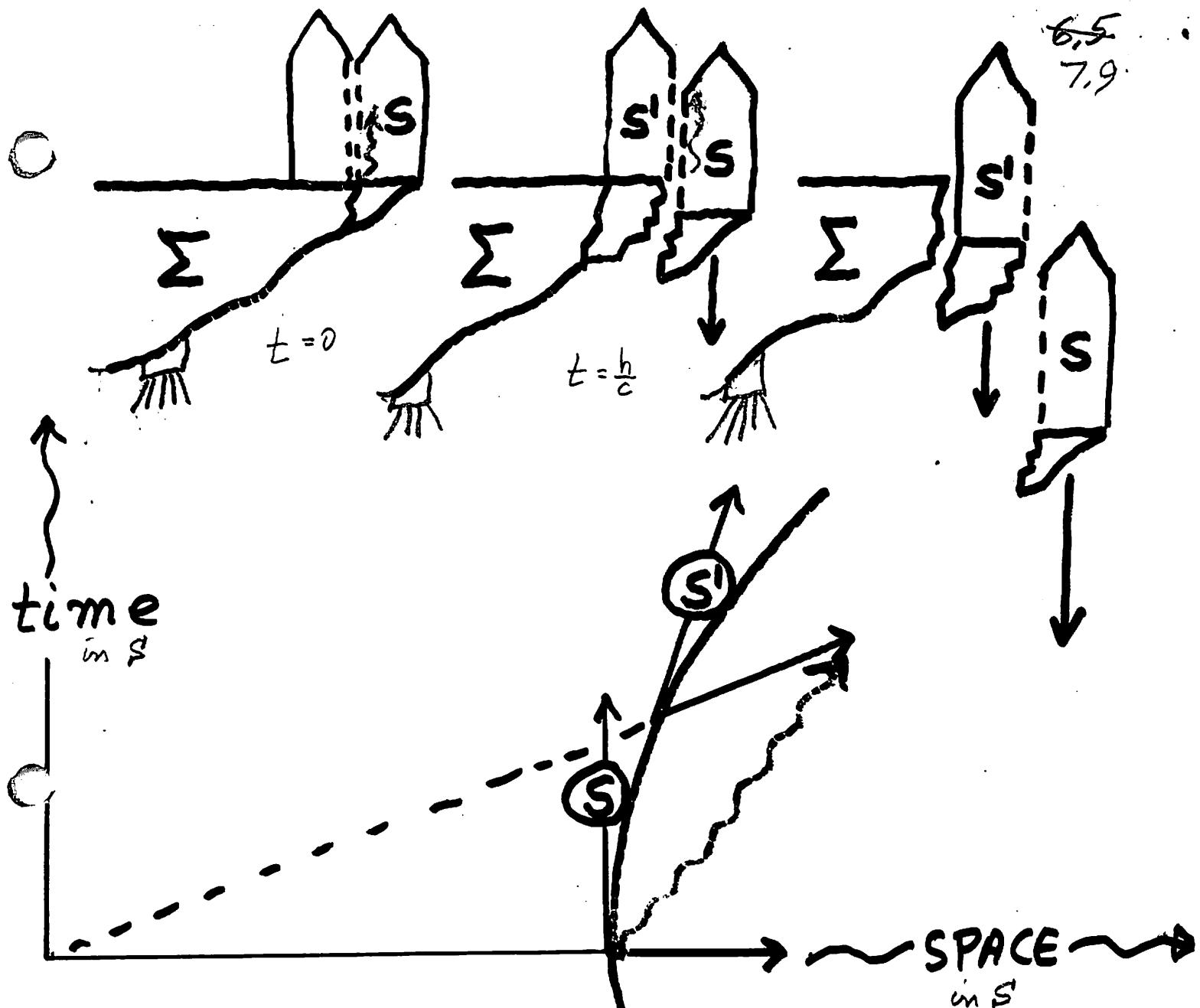
There is no difference between an uniformly accelerated frame and a frame with a static and homogeneous gravitational field:

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Regardless of their composition, the motion of bodies is the same in both frames, i.e. the two frames are equivalent (indistinguishable) no matter whether one examines the motion of bodies composed of Fe, Au, Al, ..., or even snakewood, as was done by Eötvös in his experiments.

III. Einstein's analysis consisted of introducing a pair of inertial frames S and S' and to examine the frequency of a photon relative to these frames,

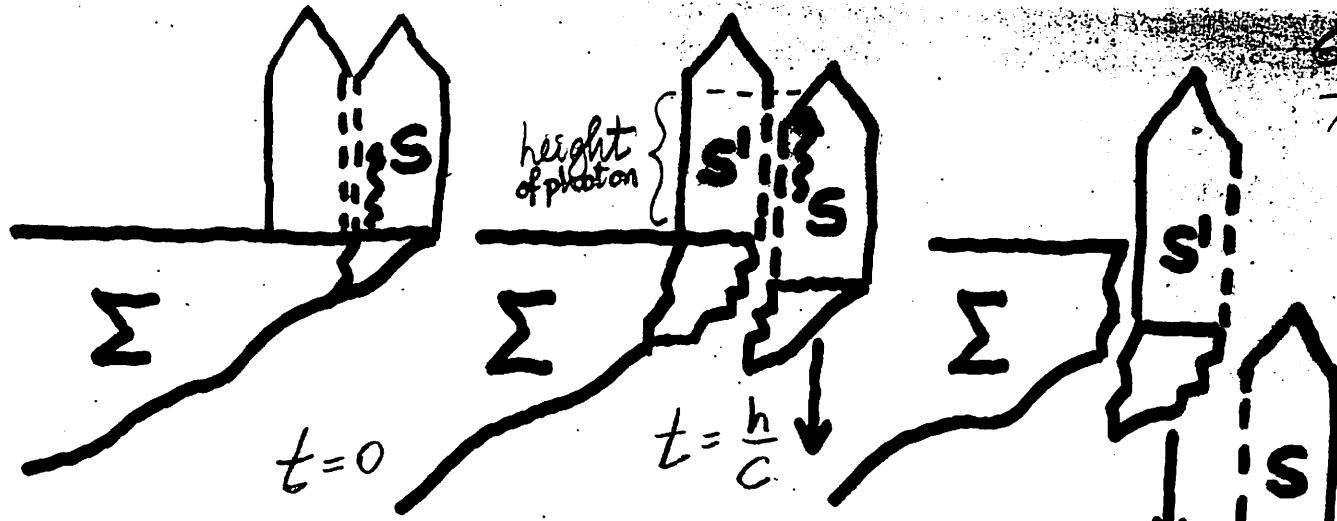
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How Einstein identified the concept of a tangent space:

C FRAGMENTIZE spacetime into locally INERTIAL FRAMES!

FIGURE 1a



When $t = \frac{h}{c}$, S' goes into free fall \Rightarrow
constant relative velocity $= \frac{h}{c} g$;

If emission freq'y relative
to S (at $t=0$) is ω_0 , what is
the freq'y relative to S'
(at $t = \frac{h}{c}$)?

time
in S

$t = \frac{h}{c}$

$$U' = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}} \right) \sim$$



$$K: (\omega_0, k_x) = (\omega_0, u_0)$$

SPACE

$$\text{relative velocity} = \left(\frac{\text{height}}{c} \right) g = \beta$$

ANSWER:

$$\omega(0) = U \cdot K = -u_0 \quad \text{freq'y relative to } S$$

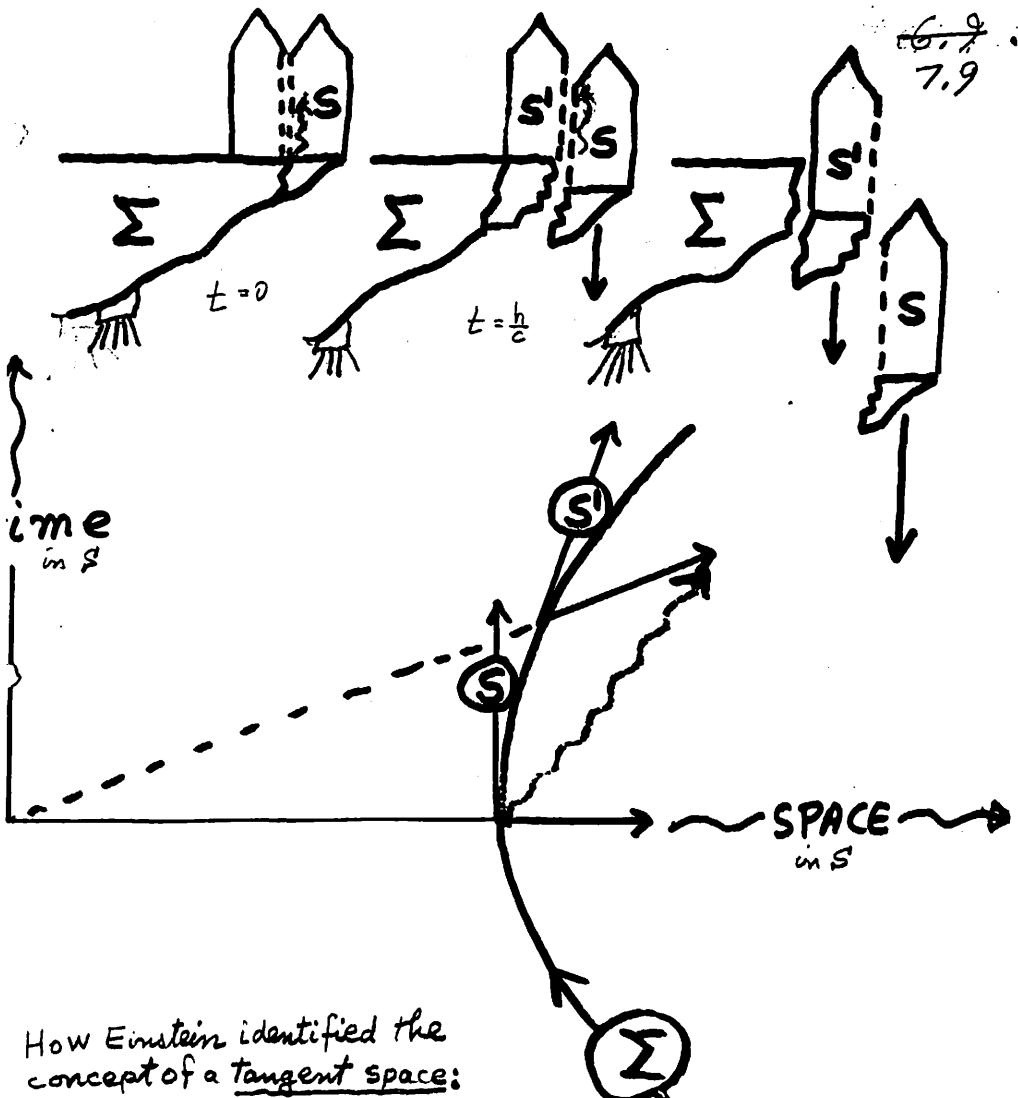
$$\omega(h) = U' \cdot K = -\frac{u_0}{\sqrt{1-\beta^2}} + \frac{u_0 \beta}{\sqrt{1-\beta^2}} \approx -u_0(1-\beta) \\ \approx -u_0(1 - \frac{hg}{c}) \quad \Sigma = \text{freq'y rel. to } S'$$

FIGURE 1b

FRAGMENTIZE spacetime into
locally INERTIAL FRAMES!

$$\omega(h) = \omega_0 \left(1 - \frac{hg}{c}\right) \quad \text{frequency as a fn of height}$$

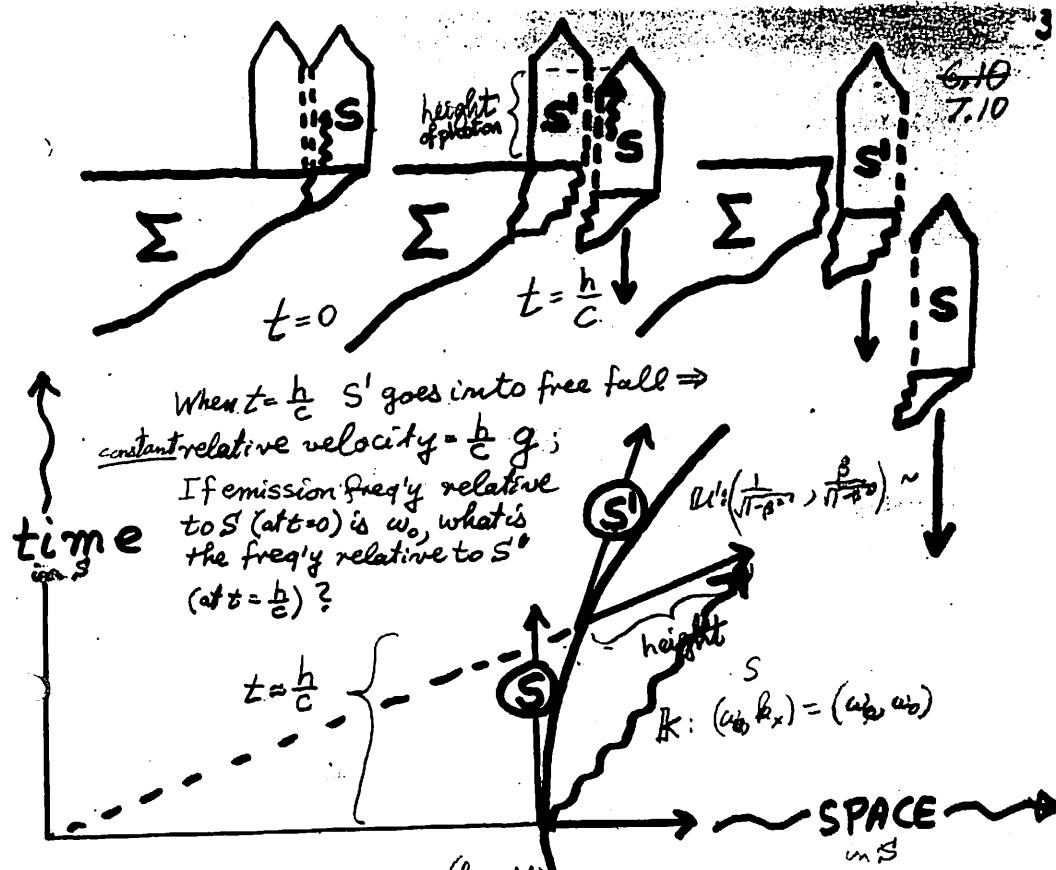
$\omega(h) - \omega(0) = "g\text{-rel. red shift}"$



How Einstein identified the concept of a tangent space:

FRAGMENTIZE Spacetime into locally INERTIAL FRAMES!

Figure 1a



$$\text{relative velocity} = \left(\frac{\text{height}}{c} \right) g = \beta$$

ANSWER:

$$\omega(0) = \omega_0 \cdot K = -\omega_0$$

$$\omega(h) = \omega_0 \cdot K = -\frac{\omega_0}{\sqrt{1-\beta^2}} + \frac{\omega_0 \beta}{\sqrt{1-\beta^2}} \approx -\omega_0(1-\beta)$$

$$\approx -\omega_0(1 - \frac{h}{c})$$

FRAGMENTIZE Spacetime into locally INERTIAL FRAMES!

$\omega(h) = \omega_0(1 - \frac{h}{c})$ frequency as a fn of height
 $\omega(h) - \omega(0) = "g \text{ grav.'l red shift.}"$

Figure 1b

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This projection method streamlines the process of calculating certain transformed vector components so much, that it permits us to give an account of how Einstein invented instantaneous Lorentz frames as the means for describing the law of nature relative to an accelerated frame.

He considers a uniformly accelerated frame Σ and two inertial frames S and S' .

At $t = 0$ S has zero velocity relative to Σ .

At that instance a photon whose four-vector is

$$K : (k^0, \vec{k}) = (\omega_0, w_0, 0, 0)$$

is launched upward. After time

$$\Delta t = \frac{h}{c}$$

when the photon has reached height h ,

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Einstein considers a second inertial frame S' with the property that at $t = \frac{h}{c}$ S' has zero velocity relative to Σ .

As a consequence the relative velocity between S and S' is

$$\beta = \frac{1}{c} \left[\frac{h}{c} \right] \text{(acceleration of } \Sigma) = \frac{h}{c^2} g$$

In other words, the accelerated frame Σ is approximated by a sequence of inertial frames

$$\Sigma : S, S', S'' \text{ etc}$$

which at successive instances where instantaneously at rest relative to Σ .

This was Einstein's key step. He immediately put it to use by calculating the gravitational redshift.

We now have the following four-velocities

$$S : u = (1, 0, 0, 0)$$

$$S' : u' = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0 \right)$$

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frequency relative to $S = -\omega \cdot K = \omega_0$

$$\text{frequency relative to } S' = -\omega' \cdot K = \frac{\omega_0}{\sqrt{1-\beta^2}} - \frac{\omega_0 \beta}{\sqrt{1-\beta^2}}$$

$$-\left(-\frac{dt'}{dx} k^0 + \frac{dx'}{dx} k^1 + \frac{dy'}{dx} k^2 + \frac{dz'}{dx} k^3\right) \approx \omega_0(1-\beta)$$

$$= \omega_0 \left(1 - \frac{hg}{c^2}\right)$$

("Doppler shifted frequency")

Einstein now equates the acceleration of Σ with the acceleration due to gravity ("equivalence principle")

Conclusion:

If at the bottom of the accelerated frame Σ

s) the photon frequency is ω_0 , then at the height h of that frame Σ the photon frequency is

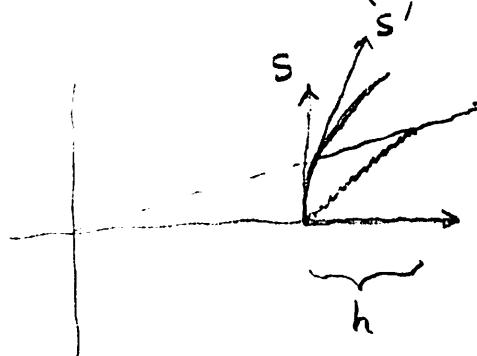
$$-\omega' \cdot K = \omega(h) = \omega_0 \left(1 - \frac{hg}{c^2}\right)$$

This is Einstein's gravitational redshift formula.

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The importance of Einstein's derivation of the gravitational red

$$\omega_{\text{rec}} = \omega_0 \left(1 - \frac{hg}{c^2}\right); \omega_0 = \text{freq'y of emitter}$$



as a Doppler shift between two inertial frames in relative motion, $v = \frac{h}{c} g$, is fundamental for mathematics;

With this derivation he introduced the concept of an instantaneous Lorentz frame, aka tangent space at each event of the spacetime history (= curve) of an arbitrarily accelerated observer Σ .

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This 1907 derivation was a break-through of the first order because it places accelerated observers within the frame work of special relativity, which is characterized by inertial (rectilinear Minkowski) frames of reference.

The domain of applicability of this derivation is

$$\beta \equiv \frac{v}{c} \equiv \frac{hg}{c^2} \ll 1.$$

What happens when $\beta \sim 1$, i.e we have large h or/and large g ?

In that case we must first determine the spacetime trajectory of a perpetually accelerated frame of reference.
This is a three step process