

Lecture 7

1. Significance and utility of the projection invariant.
2. How and why Einstein formed the concept of "instantaneous Lorentz frames"
[MTW Ch. 6]

How We Know

Work!

Knowledge is produced { discovered
learned maintained }

in the form of concepts.

• Knowledge is work

Knowledge is the goal of (objective) thinking

1. Cognition & Measurement
 2. Concept formation
 3. Abstraction from Abstractions
 4. Concept of Consciousness
 5. Definitions
 6. Axiomatic Concepts
 7. The Cognitive Role of Concepts
 8. Consciousness & Identity

Summary

To

Relativity Physics Mathematized by Geometry.

- Given:
- The emission event of a γ ray laser pulse (x-ray pulse, γ -ray burst)
 - This pulse has properties which are measured in the
 - lab frame
 - rocket frame

Question: How does one put into quantitative form, i.e. how does one mathematize

- what is measured relative to the LAB and to the ROCKET
- the relationship between these measurements?

Answer: Use the spacetime version of the Euclidean orthogonal projection between pairs of vectors

The invariants $K \cdot K$, $\Omega \cdot \Omega$, and $\bar{\Omega} \cdot \bar{\Omega}$

(i) Invariance of $K \cdot K$

(i)

$$\psi = \cos(k_x x + k_y y + k_z z - \omega t)$$

is a solution

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial t^2} = 0$$

relative
to LAB.
frame S

provided

$$-\omega^2 + k_x^2 + k_y^2 + k_z^2 = 0$$

$$-(k^0)^2 + (k^1)^2 + (k^2)^2 + (k^3)^2 = \gamma_{\mu\nu} k^\mu k^\nu$$

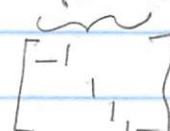


relative
to ROCKET
frame S

provided

$$-\bar{\omega}^2 + \bar{k}_x^2 + \bar{k}_y^2 + \bar{k}_z^2 = 0$$

$$-(\bar{k}^0)^2 + (\bar{k}^1)^2 + (\bar{k}^2)^2 + (\bar{k}^3)^2 = \gamma_{\bar{\mu}\bar{\nu}} \bar{k}^{\bar{\mu}} \bar{k}^{\bar{\nu}}$$



$$\text{P.of R.: } \gamma_{\mu\nu} k^\mu k^\nu = \gamma_{\bar{\mu}\bar{\nu}} \bar{k}^{\bar{\mu}} \bar{k}^{\bar{\nu}} = K \cdot K \quad (= \text{invariant})$$

(ii) Invariance of $u \cdot u$

$$\lim_{\Delta \tau \rightarrow 0} \left(\frac{1}{\Delta \tau} \right)^2 (\Delta \tau)^2 = (\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

implies

$$-1 = \gamma_{\mu\nu} \frac{dx^{\mu} dx^{\nu}}{d\tau d\tau} = \gamma_{\bar{\mu}\bar{\nu}} \frac{dx^{\bar{\mu}} dx^{\bar{\nu}}}{d\tau d\tau} = \boxed{u \cdot u = -1}$$

= (invariant)

(iii) Invariance of $u \cdot K$:

$$u \cdot K = \frac{1}{2} [(u + K) \cdot (u + K) - (u - K) \cdot (u - K)] = \gamma_{\mu\nu} u^{\mu} K^{\nu}$$

$$= \gamma_{\mu\nu} u^{\bar{\mu}} K^{\bar{\nu}}$$

is an invariant because each term
on the right is an invariant

Contracting μ and $\bar{\mu}$, or ν and $\bar{\nu}$, one can show

$$u^{\mu} u^{\bar{\mu}} - u^{\bar{\mu}} u^{\mu} = m^0 \delta^{\mu \bar{\mu}} + m^1 \delta^{\mu \bar{\mu}} - d^{\mu \bar{\mu}} = M^{\mu \bar{\mu}}$$

as can be seen from

= Iraq's theory
from the book but
without derivation.

An invariant such as $\mathbf{U} \cdot \mathbf{k}$ is important because it provides another way (besides the Lorentz transformation method) for relating the observations in different frames (i.e., "from different perspectives") of the same phenomenon. The method of invariants presupposes Lorentz transformations but it does not make explicit use of them.

Example (Planewave frequency relative to different inertial frames)

GIVEN: (1) A plane wave whose propagation four-vector has components

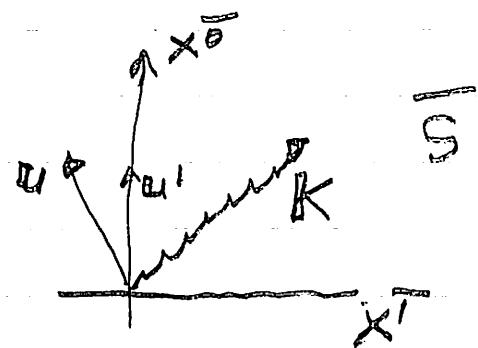
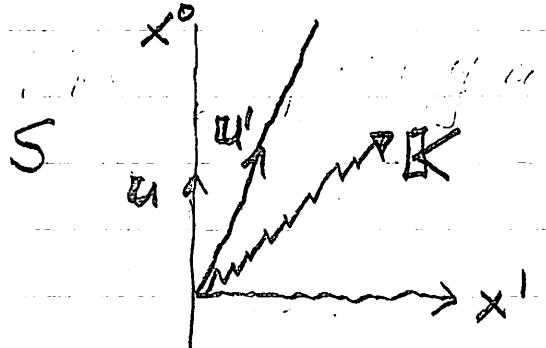
$$\mathbf{k} : \{k^0, k^1, k^2, k^3\}.$$

with frequency $\omega = k^0$, all relative to the LAB frame S,

(2). An observer in a ROCKET frame \bar{S} ,

His four-velocity components relative to the LAB frame are

$$U^i = \{U^0, U^1, U^2, U^3\}$$



$$K: \{k^0 = \omega, k^1 = \omega, k^2 = 0, k^3 = 0\}$$

$$\bar{K}^0 = \bar{\omega}, \bar{K}^1 = \omega, \bar{K}^2 = 0, \bar{K}^3 = 0$$

FIND: Relative to \bar{S} the frequency $\bar{\omega}$

of the given wave.

SOLUTION:

a) Relative to \bar{S} one has

$$U^i: \{U^0, U^1, U^2, U^3\} = \{1, 0, 0, 0\} \left(= \left\{ \frac{dx^i}{d\tau} \right\} \right)$$

$$K: \{k^0, k^1, k^2, k^3\}$$

$$u^i \cdot K = -U^0 k^0 + 0 + 0 + 0 = -k^0 = -\bar{\omega} \quad (=?)$$

$$= \underbrace{\left(\eta_{\mu\nu} U^\mu \bar{K}^\nu \text{ where } [\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right)}_{\text{where } [\eta_{\mu\nu}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

b) Relative to S one has

$$\begin{aligned} \mathbf{U}' &: \{u'^0, u'^1, u'^2, u'^3\} \left(= \left\{ \frac{dx'^M}{dt} \right\} \right) \\ \mathbf{K} &: \{k^0, k^1, k^2, k^3\} \end{aligned} \quad \left. \begin{array}{l} \text{(given)} \\ \text{(i.e. known)} \end{array} \right\}$$

$$\mathbf{U}' \cdot \mathbf{K} = -\gamma_m u'^M k^M \equiv -u'^0 k^0 + u'^1 k^1 + u'^2 k^2 + u'^3 k^3$$

c) Invariance of $\mathbf{U}' \cdot \mathbf{K}$:

$$\bar{\omega} = -\mathbf{U}' \cdot \mathbf{K} = u'^0 k^0 + u'^1 k^1 + u'^2 k^2 + u'^3 k^3$$

as calculated relative
to S.

= frequency of wave in
ROCKET frame, but
calculated relative to
the LAB frame,

d) Conclusion:

Geometrically one has

- $\bar{\omega} = \mathbf{U}' \cdot \mathbf{K}$ = projection of \mathbf{K} along
unit vector \mathbf{U}' ,

- $\omega = \mathbf{U} \cdot \mathbf{K}$ = projection of \mathbf{K} along
unit vector \mathbf{U} .

This projection technique is a shortcut
which circumvents the direct use of
a Lorentz transformation.

Accelerated Frame via
Inertial ("Free Float") Frames.

I. By examining the relation between inertial frames, an accelerated frame and a frame with a static gravitational field, we shall illustrate a type of reasoning which has opened (in this case on the part of Einstein in 1907) new vistas in physics and mathematics.

- Other examples of such reasoning are
- (i). Galileo's examination of vertical and horizontal motion
 - (ii) Newton's analysis of circular motion, which gave rise among others vector analysis

(iii) Euler's analysis of the motion
of fluids and his analysis of a vibrating
membrane.

(iv)

II. The context of Einstein's 1907 analysis
of an accelerated frame was the
observation that



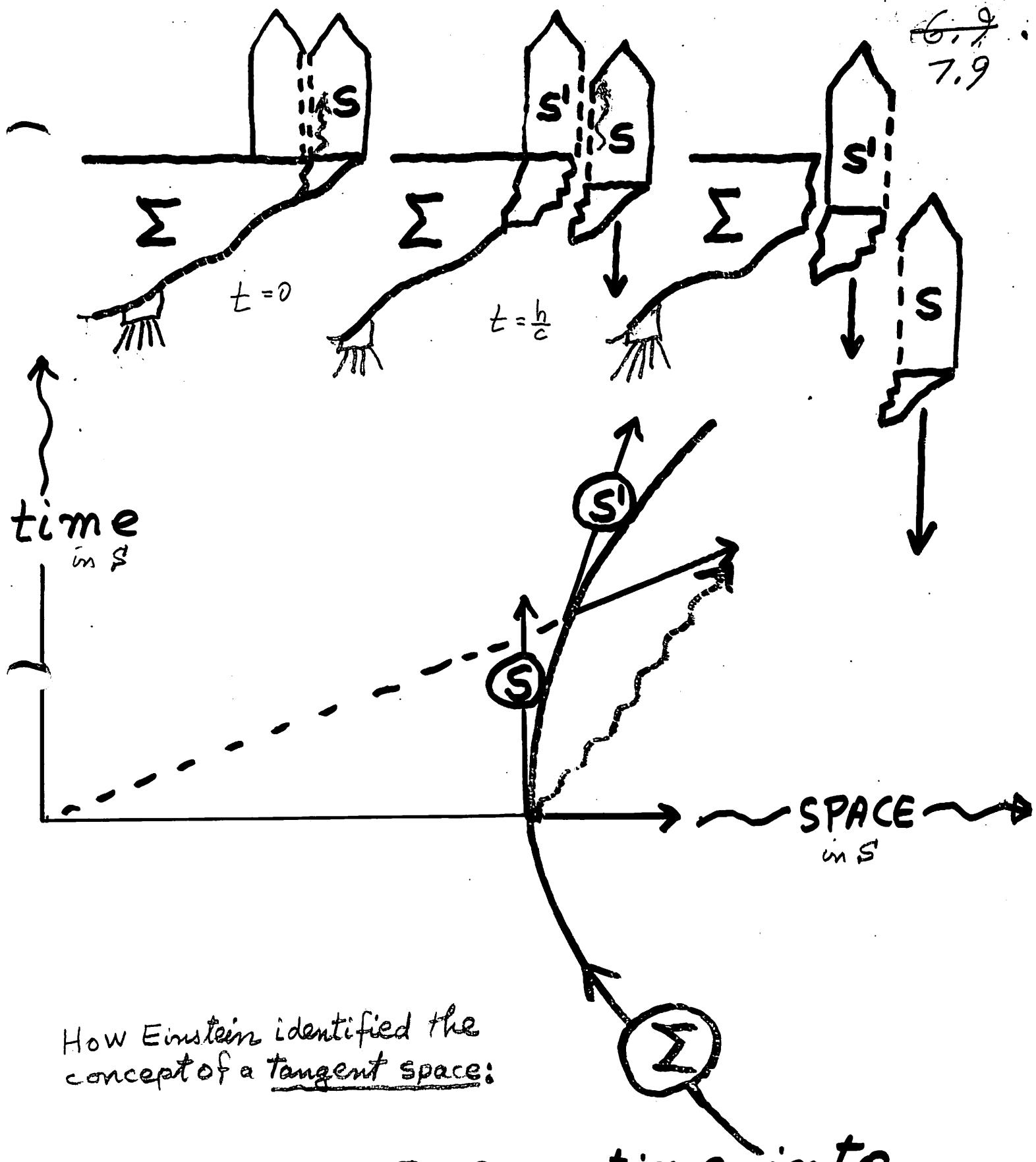
there is no difference between an
uniformly
accelerated frame and a frame with
a static and homogeneous gravitational
field.

~~6,8~~
7,8

Regardless of their composition, the motion of bodies is the same in both frames, i.e., the two frames are equivalent (indistinguishable) no matter whether one examines the motion of bodies composed of Fe, Au, Al, ..., or even snakewood, as was done by Eotvos in his experiments.

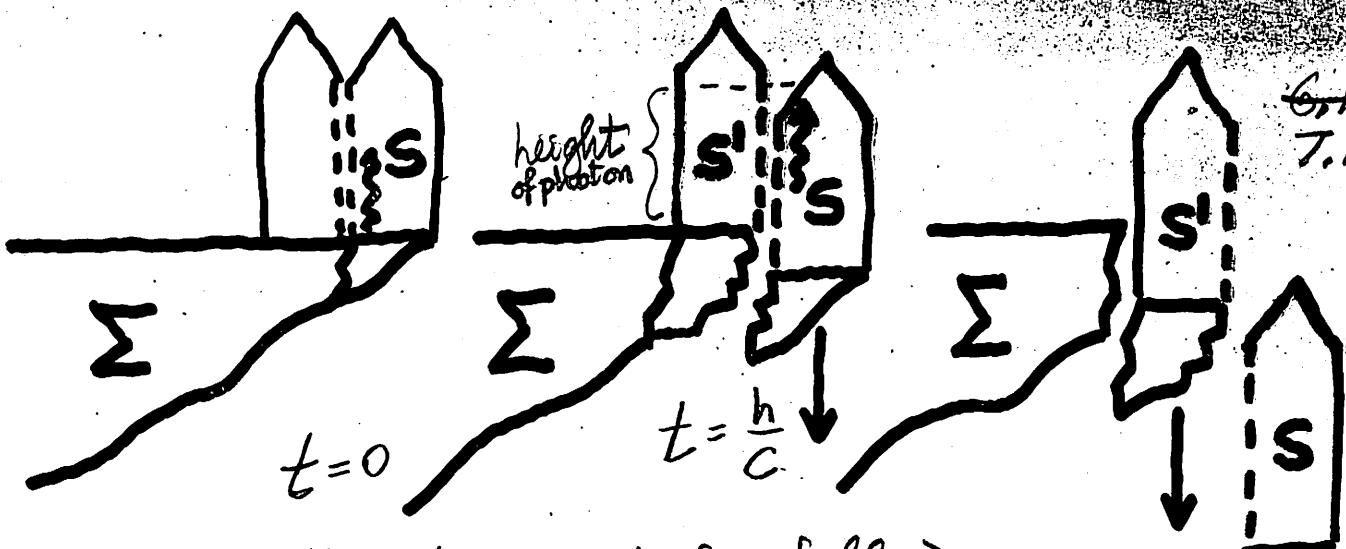
III. Einstein's analysis consisted of introducing a pair of inertial frames S and S' and to examine the frequency of a photon relative to these frames,

6.9
7.9



How Einstein identified the concept of a tangent space:

FRAGMENTIZE spacetime into locally INERTIAL FRAMES!



When $t = \frac{h}{c}$ S' goes into free fall \Rightarrow

constant relative velocity $= \frac{h}{c} g$;

If emission freq'y relative to S (at $t=0$) is ω_0 , what's the freq'y relative to S' (at $t = \frac{h}{c}$)?

time
in S

$$K' : \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}} \right) \sim$$



$$\text{relative velocity} = \left(\frac{\text{height}}{c} \right) g = \beta$$

ANSWER:

$$-\omega(0) = K \cdot K' = -\omega_0$$

freq'y relative to S

$$-\omega(h) = K' \cdot K = -\frac{\omega_0}{\sqrt{1-\beta^2}} + \frac{\omega_0 \beta}{\sqrt{1-\beta^2}} \approx -\omega_0(1-\beta)$$

$$\approx -\omega_0\left(1 - \frac{hg}{c}\right)$$

$K' = \text{freq'y rel. to } S'$

FRAGMENTIZE spacetime into
locally INERTIAL FRAMES!

$$\omega(h) = \omega_0 \left(1 - \frac{hg}{c}\right)$$

frequency as a fn of height

$$\omega(h) - \omega(0) = "g\text{-grav.'l red shift}"$$

This projection method streamlines the process of calculating certain transformed vector components so much, that it permits us to give an account of how Einstein invented instantaneous Lorentz frames as the means for describing the law of nature relative to an accelerated frame.

He considers a uniformly accelerated frame Σ and two inertial frames S and S' .

At $t = 0$ S has zero velocity relative to Σ .

At that instance a photon whose four-vector is
 $\vec{k} : (\vec{k}, \vec{k}) = (w_0, w_0, 0, 0)$
is launched upward. After time

$$\Delta t = \frac{h}{c}$$

when the photon has reached height h ,

Einstein considers a second inertial frame S' with the property that at $t = \frac{h}{c}$ S' has zero velocity relative to Σ .

As a consequence the relative velocity between S and S' is

$$\beta = \frac{1}{c} \left[\frac{h}{c} (\text{acceleration of } \Sigma) \right] \equiv \frac{h}{c^2} g$$

In other words, the accelerated frame Σ is approximated by a sequence of inertial frames

$$\Sigma : S, S', S'' \text{ etc.} \therefore \Sigma = 1\text{-param. form. of free float frames.}$$

which at successive instances were instantaneously at rest relative to Σ .

This was Einstein's key step. He immediately put it to use by calculating the gravitational redshift.

We now have the following four-velocities

$$S : u = (1, 0, 0, 0)$$

$$S' : u' = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0 \right).$$

7.13-6.13

$$\text{frequency relative to } S = -\mathbf{u} \cdot \mathbf{k} = \omega_0$$

$$\text{frequency relative to } S' = -\mathbf{u}' \cdot \mathbf{k} = \frac{\omega_0}{\sqrt{1-\beta^2}} - \frac{\omega_0 \beta}{\sqrt{1-\beta^2}}$$

$$-\left(-\frac{dt'}{dz} k^0 + \frac{dx'}{dz} k^1 + \frac{dy'}{dz} k^2 + \frac{dz'}{dz} k^3\right) \approx \omega_0(1-\beta)$$

$$= \omega_0 \left(1 - \frac{hg}{c^2}\right)$$

("Doppler shifted frequency")

Einstein now equates the acceleration of Σ with the acceleration due to gravity ("equivalence principle").

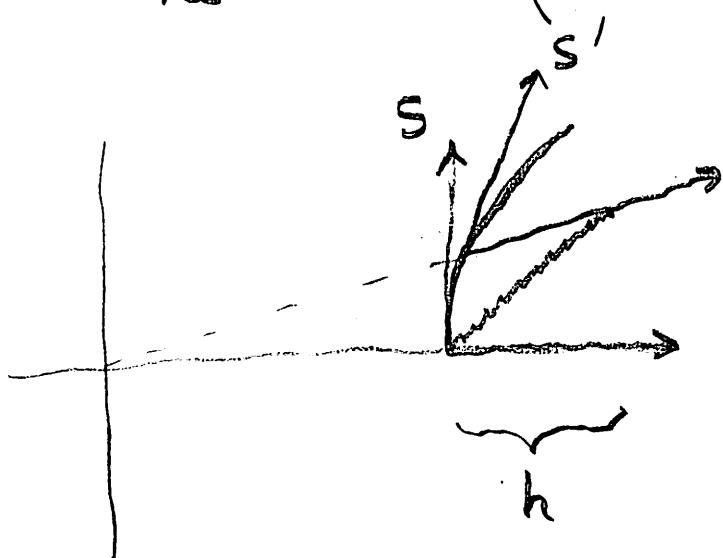
Conclusion:

If at the bottom of the accelerated frame Σ ($= S$) the photon frequency is ω_0 , then at the height h of that frame Σ \nearrow the photon frequency is
$$-\mathbf{u}' \cdot \mathbf{k} = \omega(h) = \omega_0 \left(1 - \frac{hg}{c^2}\right)$$

This is Einstein's gravitational red shift formula.

The importance of Einstein's derivation
of the gravitational red

$$\omega_{\text{rec}} = \omega_0 \left(1 - \frac{hg}{c^2}\right); \quad \omega_0 = \text{freq'y of emitter}$$



as a Doppler shift between two inertial frames in relative motion, $v = \frac{h}{c} g$, is fundamental for mathematics:

With this derivation he introduced the concept of an instantaneous Lorentz frame, a.k.a. tangent space at each event of the spacetime history (=curve) of an arbitrarily accelerated observer Σ .

~~6/15~~
7/15

- This 1907 derivation was a breakthrough of the first order because it places accelerated observers within the framework of special relativity, which is characterized by inertial (rectilinear Minkowski) frames of reference.
- The domain of applicability of this derivation is $\beta = \frac{v}{c} = \frac{hg}{c^2} \ll 1$.
What happens when $\beta \sim 1$, i.e. we have large h or/and large g ?
In that case we must first determine the spacetime trajectory of a perpetually accelerated frame of reference.
This is a three step process.