Lecture 9

Thomas Precession TW [p169-174, 1st ed.]
MTW [exercise 6.9]
Fermi-Walker xport MTW [Ch. 6]
Recap of Lecture 8

The world line
\[ x(t) = g^{-1} \cosh g t. \]
\[ x'(t) = g^{-1} \sinh g t. \]

of a uniformly accelerated body has associated with it a parametrized family of pairs of vectors, namely
\[ u(t) = \frac{dx}{dt} = \cosh g t, \quad u(t) = \frac{dx'}{dt} = \sinh g t. \]
and
\[ \alpha(t) = \frac{du}{dt} = g \sinh g t, \quad \alpha(t) = \frac{dx'}{dt} = g \cosh g t. \]

This newly defined concept compels us into the following generalization:

An accelerated frame = 1-parameter family of instantaneous inertial frames which are related by Lorentz transformations, a la "Lorentz boosts."

Comment:

1. Elie Cartan calls the 1-parameter family of vectors \( \{ u(t), \alpha(t) \} \) a "repere mobile," a moving frame.

2. The statement of this generalization lacks mathematical precision because we have as yet not defined mathematically what is meant by an "accelerated frame." We shall remedy this defining in Lecture 10.

3. The hyperbolic world line of \( \Sigma \) was based on the physical circumstance where the spatial acceleration was colinear with the spatial velocity.

4. We shall now consider the circumstance where the spatial (centripetal) acceleration is perpendicular to the spatial velocity as one has it for the case of spatially circular motion.
Our special relativity analysis of this circumstance leads us to an explanation of the Thomas precession of a gyroscope in a circular orbit.
Suppose one takes a freely gimbaled gyroscope and accelerates it by moving it along, say, a circular trajectory.

Q: What is the gyroscope's orientation after one revolution?

A: If the motion along the circular

orbit is counter-clockwise, and occurs with speed \( \beta \), then the precession angle is clockwise by an amount

\[
\phi = 2\pi(y-1) \quad y = \frac{1}{\sqrt{1-\beta^2}}
\]

Thus, if \( \omega = \frac{\phi}{t} \) is the angular velocity, the precession rate is

\[
\omega = \frac{\phi}{t} = 2\pi(y-1) \quad \text{clockwise}
\]

One obtains a qualitative understanding of this result by approximating the circle by a circumscribed polygon.

The gyroscope is then viewed as hopping from one inertial frame to another.

What is the orientation of \( B \) relative to the L.A.B frame, \( B \) because \( B \) is moving very slowly relative to \( A \).
Consider the special case of A and B horizontal relative to the instantaneous comoving frame.

\[ \overset{\sim}{\text{In } \overset{\sim}{S}:} \quad \frac{1}{\overset{\sim}{y}}: B, \quad B \overset{\text{drifting}}{\rightarrow} \overset{\sim}{y} \]

\[ \overset{\sim}{A} \quad \overset{\sim}{\text{LAB motion}} \]

B's orientation relative to LAB frame is no longer horizontal because two events simultaneous in comoving frame are no longer simultaneous in LAB frame.

\[ \overset{\sim}{\text{In } \overset{\sim}{S}:} \quad \frac{1}{\overset{\sim}{y}}: B, \quad B_2 \quad \overset{\text{drifting}}{\rightarrow} \overset{\sim}{y} \]

\[ \overset{\sim}{A} \quad \overset{\sim}{\text{LAB motion}} \]

Given: B, B_2 are two events simultaneous in the comoving frame.

Q: In the LAB frame, which event occurs earlier, B, or B_2?

A: B, is earlier than B_2.

\[ \text{In LAB frame } B, \text{ is above } B_2, \text{i.e. gyro is rotated clockwise!} \]

Q: What is the reasoning leading to this conclusion?

A: The light signals from B, and B_2 reach the spatial origin \( x=0 \) of \( \overset{\sim}{S} \) at the same time \( \overset{\sim}{E} \).

So that \( \overset{\sim}{S} \) measures B, and B_2 to occur at \( \overset{\sim}{E} = \overset{\sim}{E}_1 = \overset{\sim}{E}_2 = 0 \).

By contrapositive, \( S \) records \( t_1 < t_2 \):

\[ t_1 - t_2 > 0 \]

Thus the y motion of B relative to S is:

\[ y = y_1 \]

\[ (t_2 - t) = y_2 \]

\[ y = y_1 \]

\[ y = y_1 \]
In the LAB frame event B occurs before event B₂. Thus due to the upward motion of B, B₂ side of the gyroscope crosses a horizontal line before the B₂ side of that gyroscope does.

As seen in the LAB frame

Conclusion: LAB observer records A → B to have undergone a clockwise rotation, i.e.

a freely gimbaled gyroscope precesses into a clockwise direction relative to the fixed stars whenever its orbital motion is counterclockwise.
First Comment: The relativity of simultaneity about events B, and B₂ can also be applied to the direction of a moving wave pattern.

Consider two wave patterns relative to a given inertial frame: one moving strictly upward, the other downward.

```
\[ \begin{array}{c}
\text{upward moving} \\
\text{wave pattern}
\end{array} \quad \begin{array}{c}
\text{downward moving} \\
\text{wave pattern}
\end{array} \]
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Question: Suppose the "LAB" frame has a leftward motion relative to the given frame:

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\[ \text{LAB frame} \quad \text{given frame} \]
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What is the direction of the two moving wave patterns relative to the LAB frame?

Answer:

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\[ \begin{array}{c}
\text{B}_1 \quad \text{B}_2
\end{array} \]
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Conclusion: The faster the "LAB" frame is moving relative to the given frame, the smaller an angle do their wave propagation vectors (normals to the phase fronts) make with the positive horizontal axis.

Second comment (The Bohr Ruler)

Question: What is the ratio \( \frac{\pi}{2} \) for an electron in its first Bohr orbit of the hydrogen atom?

Answer:

\[ \frac{\nu}{c} = \frac{e^2}{\hbar c} = \frac{1}{137} = \alpha = \left( \text{the fine structure constant} \right) \]

where \( e \) = the electron charge, \( \hbar \) = Planck's constant, \( c \) = speed of light, and \( \alpha \) = incommensurable units.
Consider the three fundamental lengths associated with a charged mass:

- Its Bohr (orbital) radius: $a_B = \alpha a_B$
- Its Compton wavelength: $\lambda_c = \frac{\hbar}{mc}$ (radius)
- Its classical electron radius: $\tau_e = \frac{e^2}{mc^2}$

One has the following relationships:

\[
a_B = a_B \times 1; \quad \lambda_c = a_B \times \alpha; \quad \tau_e = a_B \times \alpha^2
\]

or

\[
a_B = \frac{\hbar}{mc}, \quad \lambda_c = \frac{\hbar}{mc}, \quad \tau_e = \frac{\hbar}{mc} \times \frac{e^2}{mc^2}
\]

These three lengths make up the Bohr ruler:

\[
\begin{array}{ccc}
\hbar^2/mc^2 & \hbar/mc & e^2/mc^2 \\
1.529 \times 10^{-8} \text{ cm} & 3.86 \times 10^{-11} \text{ cm} & 2.8 \times 10^{-13} \text{ cm}
\end{array}
\]

Bohr radius | Compton wavelength | Classical electron radius

These are the lengths for an electron whose charge $e = 4.8 \times 10^{-10} \text{ (g cm}^3\text{)}^{-\frac{1}{2}}$ + "stat coulomb" and mass $m = 9 \times 10^{-28} \text{ gr}$ (all in c.g.s. units)
Appendix to Lecture 9


"Why does an orbiting gyroscope precess?"

The observation that a gyroscope does not precess in the comoving frame needs elaboration. For illustrative purposes consider a classical spinning particle orbiting around an attractive center.

Approximate the circular motion by the motion around a circumscribed polygon.

We do this so that we can take the limit of this discrete family of Lorentz frames and thereby recover the continuous 1-parameter family associated with the given circular motion.

2. Question: As the spinning particle moves from one Lorentz frame to another, how does its spin direction change? (a) as seen in the LAB, and (ii) as seen in the COMOVING ("roded") frame.

In the comoving frame the particle is in two consecutive states of motion, A and B.
3. Answer:

We now say that the spin orientation changes in the same way as a meter stick in the comoving frame.

A looks at B and finds AB
B looks at A and finds BA
Both see parallelness because their relative velocity is small (but non-zero)
This parallelism, as the particle changes its state of motion from A to B, is the essence of the Fermi–Walker transport.

\textit{i.e., no rotation of the spin vector if the particle goes from state A to state B.}

4. Question: If the spin vector \( \vec{P} \)

\[ \begin{array}{c}
A \quad \text{[Diagrams of A and B with spin vectors]} \\
B \quad \text{[Diagram of B with spin vector]} \\
\text{"drifting vector } \vec{P} \" \\
\end{array} \]

does not change as the state of motion of the particle changes from \( A \) to \( B \) relative to the comoving frame, will the same observation hold relative to the LAB frame?

Answer: No.

Why?
5. Why?

The relativity of simultaneity demands that the two events P and Q, which marked the simultaneous crossing of the x-axis by the drifting (upward) vector PQ relative to the comoving frame, do not mark a simultaneous crossing of the x-axis relative to the LAB frame. Unlike a comoving observer who remains halfway between P and Q,

\[ A \rightarrow B \]

\[ \Delta t = \text{LAB time to move from A to B.} \]

\[ \Omega = \text{angular velocity} \]

6. What is this angle of rotation?

a) This angle is small because we are considering two inertial motions A and B, which differ only slightly.
In order to continue the quantitative development, one must establish the relationship between three different inertial frames:

(i) the LAB frame
(ii) the COMOVING inertial frame A,
(iii) the COMOVING inertial frame B.

Index notation, which refers directly to the frame components of a vector, is very efficient and hence useful is one is confronted with only one or two different frames. However, when there are three or more frames, then the matrix notation enjoys considerable superiority.

We therefore interrupt the development with three tutorials which concern the index and matrix notation and the application of the latter to express a change in the representation of a linear transformation.