

# Lecture # 9

Thomas Precession TW [CP169-174, 1<sup>st</sup> ed],

MTW [exercise 6.9]

Fermi-Walker xport MTW [ch. 6]

## Recap of Lecture 8

The world line

$$x^0(\tau) = g^{-1} \text{sh } g\tau$$

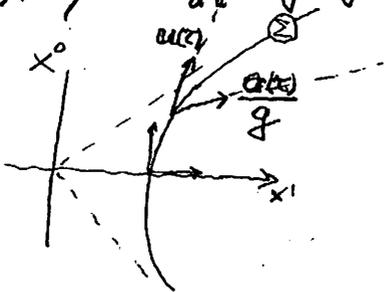
$$x^1(\tau) = g^{-1} \text{ch } g\tau$$

of a uniformly accelerated body has associated with it a parametrized family of pairs of vectors, namely

$$u(\tau): \{ u^0(\tau) \equiv \frac{dx^0}{d\tau} = \text{ch } g\tau, u^1(\tau) \equiv \frac{dx^1}{d\tau} = \text{sh } g\tau \}$$

and

$$a(\tau): \{ a^0(\tau) \equiv \frac{du^0}{d\tau} = g \text{sh } g\tau, a^1(\tau) \equiv \frac{du^1}{d\tau} = g \text{ch } g\tau \}$$



This newly defined concept compels us into the following generalization:

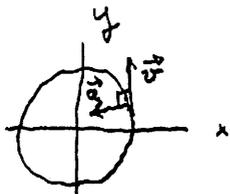
An accelerated frame = 1-parameter family of instantaneous inertial frames which are related by Lorentz transformations, aka "Lorentz boosts"

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9.0a

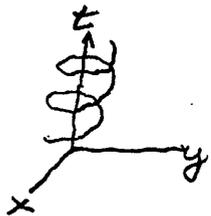
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~~8.06~~

Comment:

- (1) Elie Cartan calls the 1-parameter family of vectors,  $\{ u(\tau), \frac{a(\tau)}{g} \}$  a "repere mobile", a moving frame.
- (2) The statement of this generalization lacks mathematical precision because we have as yet not defined mathematically what is meant by an "accelerated frame." We shall remedy this deficiency in Lecture 10.
- (3) The hyperbolic worldline of  $\epsilon$  was based on the physical circumstance where the spatial acceleration was colinear with the spatial velocity.
- (4) We shall now consider the circumstance where the spatial (centripetal) acceleration is perpendicular to the spatial velocity as one has it for the case of spatially circular motion.



spatial  
orbit  
of a body



spacetime  
worldline  
of that body

~~9.00~~  
~~8.00~~

Our special relativity analysis  
of this circumstance leads us to an  
explanation of the Thomas precession  
of a gyroscope in a circular orbit.

Suppose one takes a freely gimballed gyroscope and accelerates it by moving it along, say, a circular trajectory.

Q: What is the gyroscope's orientation after one revolution?

A: If the motion along the circular



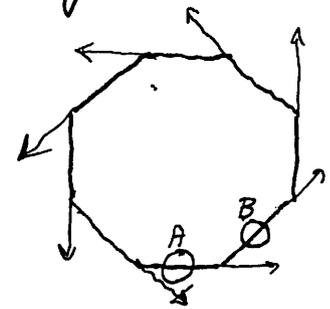
orbit is counter clockwise, and occurs with speed  $\beta$ , then the precession angle is clockwise by an amount

$$\phi = 2\pi(\gamma - 1) \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

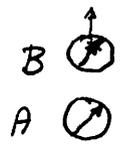
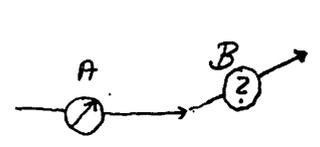
Thus, if  $\Omega = \frac{2\pi}{T}$  is the angular velocity, the precession rate is

$$\omega = \frac{\phi}{T} = \Omega(\gamma - 1) \quad \text{clockwise}$$

One obtains a qualitative understanding of this result by approximating the circle by a circumscribed polygon



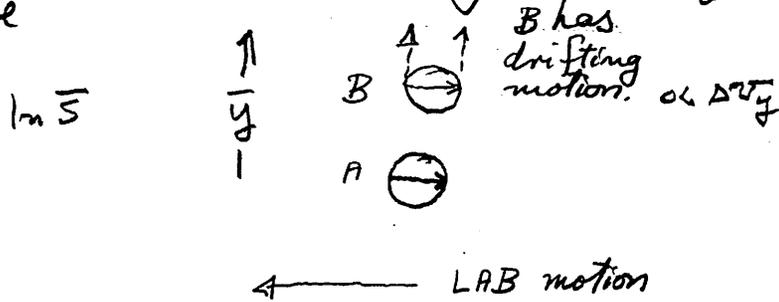
The gyroscope is then viewed as hopping from one inertial frame to another.



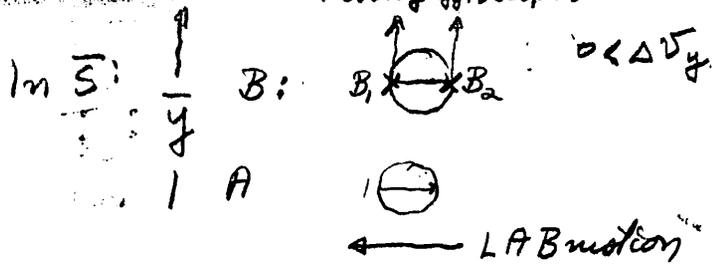
What is the orientation of B relative to the LAB frame? (Fixed stars)

What the comoving observer sees: A || B because B is moving very slowly relative to A.

Consider the special case of A and B horizontal relative to the instantaneously comoving frame



B's orientation relative to LAB frame is no longer horizontal because two events simultaneous in comoving frame are no longer simultaneous in LAB frame:  
 Moving gyroscope:



Given: B<sub>1</sub> & B<sub>2</sub> are two events simultaneous in the comoving frame

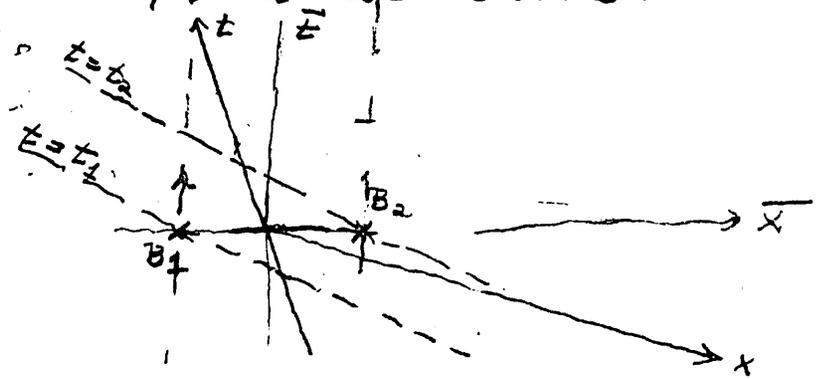
Q: In the LAB frame, which event occurs earlier B<sub>1</sub> or B<sub>2</sub>?

A: B<sub>1</sub> is earlier than B<sub>2</sub>.

∴ In LAB frame B<sub>1</sub> is above B<sub>2</sub>, i.e. gyro is rotated clockwise!

Q: what is the reasoning leading to this conclusion?

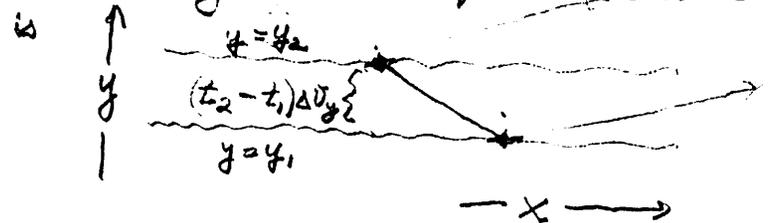
A: The light signals from B<sub>1</sub> and B<sub>2</sub> reach the spatial origin  $\bar{x}=0$  of  $\bar{S}$  at the same time  $\bar{t}$



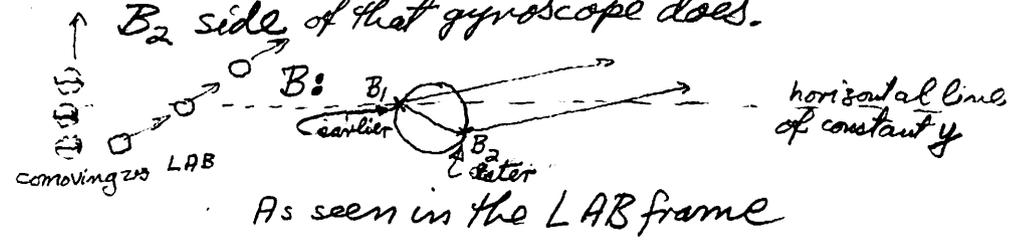
so that  $\bar{S}$  measures B<sub>1</sub> and B<sub>2</sub> to occur at  $\bar{t} = \bar{t}_1 = \bar{t}_2 = 0$ .

By contrast, observer S records  $t_1 < t_2$ :  
 $t_2 - t_1 > 0$

Thus the y motion of B relative to S



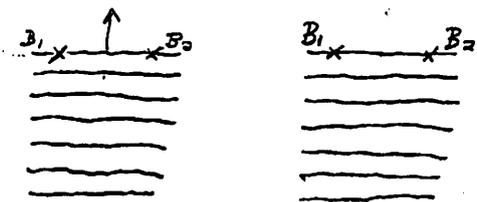
In the LAB frame event  $B_1$  occurs before event  $B_2$ . Thus due to the upward motion of  $B_1$ ,  $B_1$  side of the gyroscope crosses a horizontal line before the  $B_2$  side of that gyroscope does.



Conclusion: LAB observer records  $A \rightarrow B$  to have undergone a clockwise rotation, i.e. a freely gimbaled gyroscope precesses into a clockwise direction relative to the fixed stars whenever its orbital motion is counterclockwise.

First Comment: The relativity of simultaneity<sup>-5-</sup> about events  $B_1$  and  $B_2$  can also be applied to the direction of a moving wave pattern.

Consider two wave patterns relative to a given inertial frame; one moving strictly upward, the other downward



upward moving wave pattern

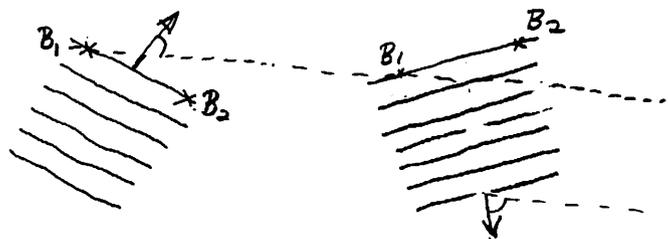
downward moving wave pattern

Question: Suppose the "LAB" frame has a leftward motion relative to the given frame:



what is the direction of the two moving wave patterns relative to the LAB frame?

Answer:



Conclusion: The faster the "LAB" frame is moving relative to the given frame, the smaller an angle do their wave propagation vectors (normals to the phase fronts) make with the positive horizontal axis.

Second comment (The "Bohr Ruler")

Question: What is the ratio  $v/c$  for an electron in its first Bohr orbit of the hydrogen atom?

Answer:  $\frac{v}{c} = \frac{e^2}{\hbar c} = \frac{1}{137} = \alpha =$  ("fine structure constant")

where  $e =$  the electron charge } incommensurable  
 $\hbar =$  Planck's constant } units.  
 $c =$  speed of light

# The Bohr Ruler

Consider the three fundamental lengths associated with a charged mass:

Its Bohr (orbital) radius =  $a_B$

Its Compton wave length =  $\lambda_c$   
(radius)

Its classical electron radius =  $r_e$

One has the following relationships

$$a_B = a_B \times 1 ; \lambda_c = a_B \times \alpha ; r_e = a_B \times \alpha^2$$

or

$a_B = \frac{h}{m c} \frac{1}{\frac{e^2}{h c}} = \frac{h^2}{m e^2}$	$\lambda_c = \frac{h}{m c}$	$r_e = \frac{h}{m c} \frac{e^2}{h c} = \frac{e^2}{m c^2}$
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These three lengths make up the Bohr ruler:

$h^2/m e^2$	$h/m c$	$e^2/m c^2$
$5.29 \times 10^{-8} \text{ cm}$	$3.86 \times 10^{-11} \text{ cm}$	$2.8 \times 10^{-13} \text{ cm}$
Bohr radius	Compton wave length	classical electron radius

These are the lengths for an electron whose charge  $e = 4.8 \times 10^{-10} \left( \frac{\text{gr cm}^3}{\text{sec}^2} \right)^{1/2}$  + "stat coulomb"

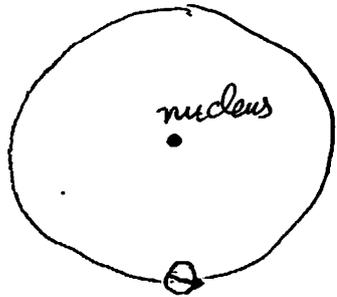
and mass  $m = 9 \times 10^{-28} \text{ gr}$   
(all in c.g.s units)

# Appendix to Lecture 9

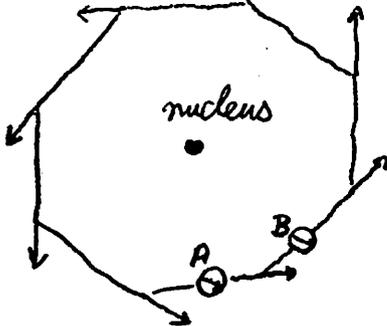
① Thomas Precession: Its Essentials. -1-

"Why does an orbiting gyroscope precess?"

The observation that a gyroscope does not precess in the comoving frame needs elaboration. For illustrative purposes consider a classical spinning particle orbiting around an attractive center



Approximate the circular motion by the motion around a circumscribed polygon

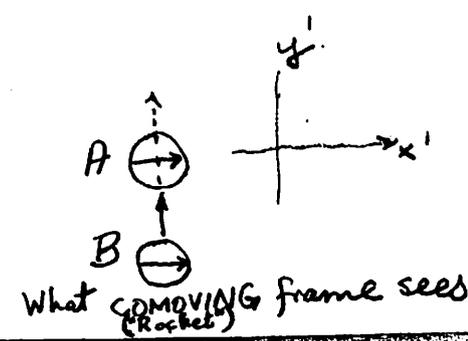
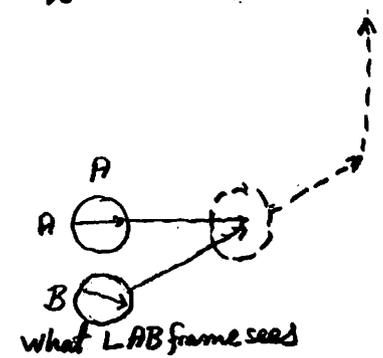


-2-

We do this so that we can take the limit of this discrete family of Lorentz frames and thereby recover the continuous 1-parameter family associated with the given circular motion.

2. Question: As the spinning particle moves from one Lorentz frame to another, how does its spin direction change (as seen in the LAB, and (ii) as seen in the COMOVING ("rocket") frame.

In the comoving frame the particle is in two consecutive states of motion, A and B.



3. Answer:

We now say that the spin orientation changes in the same way as a meter stick in the comoving frame.

A looks at B and finds  $A \parallel B$ .

B looks at A and finds  $B \parallel A$

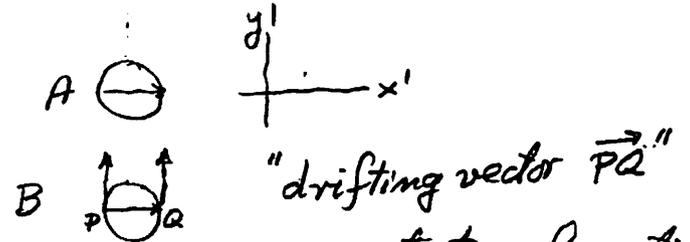
Both see parallelness because their relative velocity is small (but non-zero)

This parallelism, as the particle changes its state of motion from A to B, is the essence of the Fermi-Walker transport.

i.e. no rotation of the spin vector <sup>(ab)</sup> the particle goes from state A to state B.

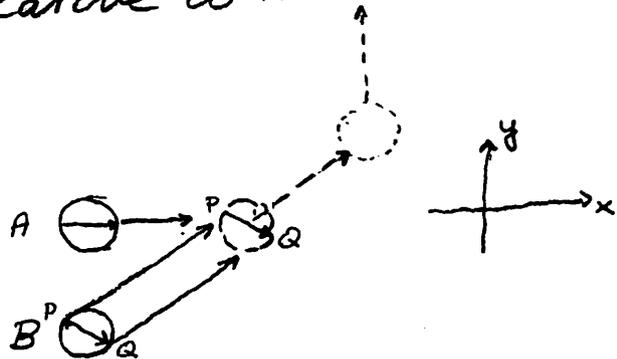
-3-

4. Question: If the spin vector  $\vec{PQ}$



does not change as the state of motion of the particle changes from A to B relative to the COMOVING frame, will

the same observation hold relative to the LAB frame?



Answer: No.

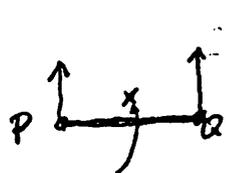
Why?

-4-

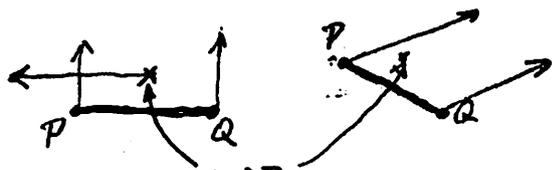
5. Why?

-5-

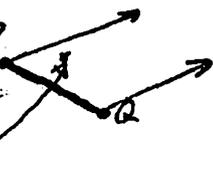
The relativity of simultaneity demands that the two events P and Q, which marked the simultaneous crossing of the  $x'$ -axis by the drifting (upward) vector PQ relative to the COMOVING frame, do not mark a simultaneous crossing of the  $x$ -axis  $(t=0)$  relative to the LAB frame. Unlike a COMOVING observer who remains halfway between P and Q,



COMOVING observer



LAB observer  
As seen in COMOVING frame (moves to the left.)



As seen in LAB frame (does not move at all.)

-6-  
the LAB observer moves towards P and recedes from Q. Consequently event P will be observed before event Q by this LAB observer. Since PQ is moving upward, P crosses the  $x$ -axis before Q does, i.e. PQ is rotated clockwise relative to the  $x$ -axis of the LAB frame.

6. What is this angle of rotation?  
This angle is small because we are considering two inertial motions A and B, which differ only slightly.



$$\alpha = \frac{2\pi}{T} = \Omega \Delta t$$

$\Delta t$  = LAB time to move from A to B.  
 $\Omega$  = angular velocity

In order to continue the quantitative development, one must establish the relationship between three different inertial frames:

- (i) the LAB frame
- (ii) the COMOVING inertial frame A,
- (iii) the COMOVING inertial frame B.

Index notation, which refers directly to the frame components of a vector, is very efficient and hence useful is one is confronted with only one or two different frames. However, when there are three or more frames then the matrix notation enjoys considerable superiority.

We therefore interrupt the development with three tutorials which concern the index and matrix notation and the application of the latter to express a change in the representation of a linear transformation.