

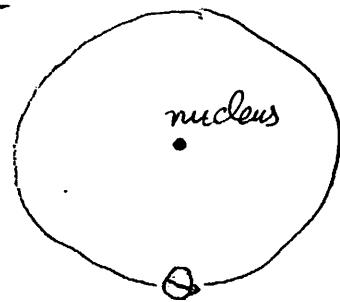
SUPPLEMENT TO LECTURE 9

- ① Thomas Precession: Its Essentials
[Exercise 103 P169-174 in T & W (1st, original edition)
1966]
- ② Tutorial I: Index vs Matrix Notation (T.1-T.6)
[Box 2, 4 and Section 2.9 in MTW]
- Tutorial II: Index vs Matrix Representation
of a Transformation. (T.7-T.10)
- Tutorial III: Change of Representation
(T.11-T.14)
- ③ Thomas Precession: Quantitative Determination
(1-14)

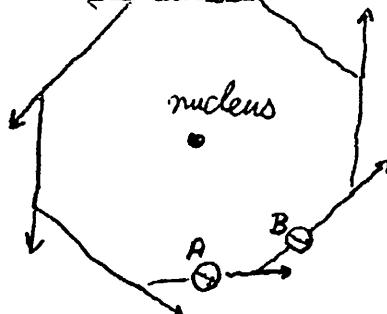
① Thomas Precession : Its Essentials.

-1-

The observation that a gyroscope does not precess in the comoving frame needs elaboration. For illustrative purposes consider a classical spinning particle orbiting around an attractive center



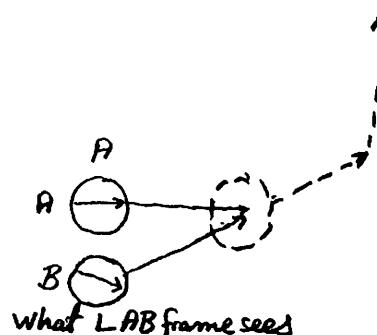
Approximate the circular motion by the motion around a circumscribed polygon



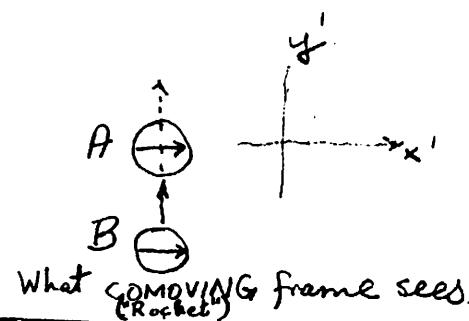
We do this so that we can take the limit of this discrete family of Lorentz frames and thereby recover the continuous 1-parameter family associated with the given circular motion.

2. Question: As the spinning particle moves from one Lorentz frame to another, how does its spin direction change (i) as seen in the LAB, and (ii) as seen in the COMOVING ("rocket") frame.

In the comoving frame the particle is in two consecutive states of motion, A and B.



What LAB frame sees



What COMOVING frame sees
(Rocket)

3. Answer:

We now say that the spin orientation changes in the same way as a meter stick in the comoving frame.

A looks at B and finds $A \parallel B$.

B looks at A and finds $B \parallel A$

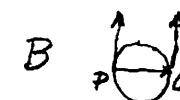
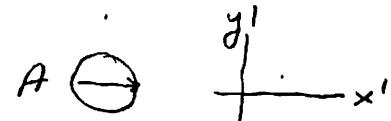
Both see parallelness because their relative velocity is small,

This parallelism, as the particle changes its state of motion from A to B, is the essence of the Fermi-Walker transport.

i.e. no rotation of the spin vector the particle goes from state A to state B.

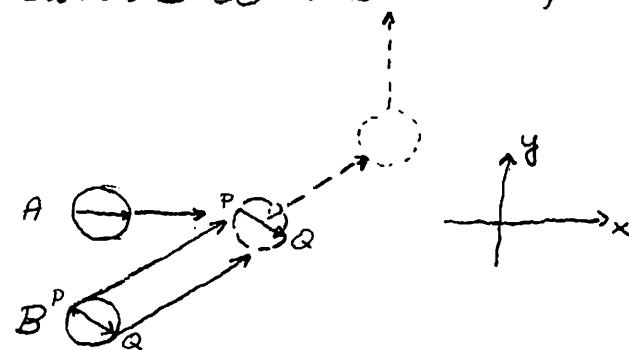
-3-

4. Question: If the spin vector \vec{PQ}



"drifting vector \vec{PQ} "

does not change as the state of motion of the particle changes from A to B relative to the COMOVING frame, will the same observation hold relative to the LAB frame?

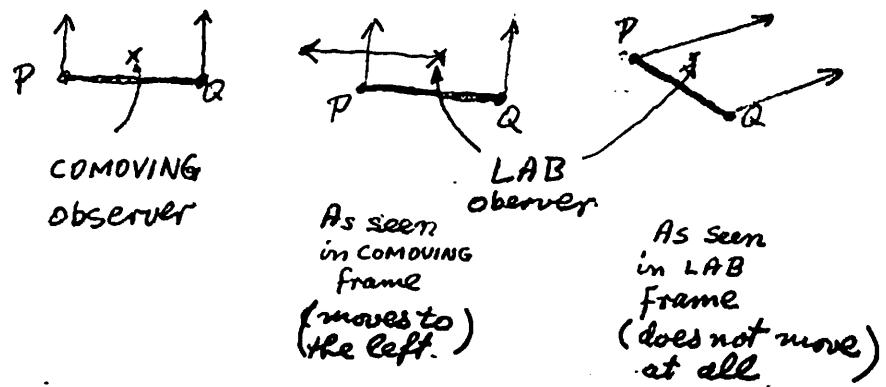


Answer: No.

Why?

5. Why?

The relativity of simultaneity demands that the two events P and Q, which marked the simultaneous crossing of the x'-axis by the drifting (upward) vector PQ relative to the COMOVING frame, do not mark a simultaneous crossing of the x-axis relative to the LAB frame. Unlike a comoving observer who remains halfway between P and Q



-5-

⁻⁶⁻ the LAB observer moves towards P and recedes from Q. Consequently event P will be observed before event Q by this LAB observer. Since PQ is moving upward, P crosses the x-axis before Q does, i.e. PQ is rotated clockwise relative to the x-axis of the LAB frame.

6. What is this angle of rotation?

- a) This angle is small because we are considering two inertial motions A and B, which differ only slightly.



$$\alpha = \frac{2\pi}{n} = \Omega \Delta t$$

Δt = LAB time to move from A to B.

Ω = angular velocity

In order to continue the quantitative development, one must establish the relationship between three different inertial frames:

- (i) the LAB frame
- (ii) the COMOVING inertial frame A.
- (iii) the COMOVING inertial frame B.

Index notation, which refers directly to the frame components of a vector, is very efficient and hence useful if one is confronted with only one or two different frames. However, when there are three or more frames then the matrix notation enjoys considerable superiority.

We therefore interrupt the development with three tutorials which concern the index and matrix notation and the application of the latter to express a change in the representation of a linear transformation.

2) TUTORIAL I: INDEX AND MATRIX Notation

Consider a spacetime displacement - T.1 -

vector \mathbf{x} . In terms of two given bases

$$\{\mathbf{e}_\alpha\} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \quad \text{"LAB Frame"}$$

and

$$\{\mathbf{e}_{\alpha'1}\} = \{\mathbf{e}'_0, \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} \quad \text{"ROCKET Frame"}$$

the same vector \mathbf{x} has the two alternate expansions.

$$\mathbf{x} = \mathbf{e}_\alpha x^\alpha = \mathbf{e}_{\alpha'1} x^{\alpha'1} \quad (1)$$

This gives rise to the two corresponding representations of \mathbf{x}

$$\{\mathbf{e}_\alpha\}: \quad \mathbf{x}^\alpha \leftrightarrow [\mathbf{x}] = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad \begin{array}{l} \text{"LAB frame"} \\ \text{representation} \end{array} \quad (2)$$

index notation \leftrightarrow matrix notation

$$\{\mathbf{e}_{\alpha'1}\}: \quad x^{\alpha'1} \leftrightarrow [\mathbf{x}]' = \begin{bmatrix} x^{01} \\ x^{11} \\ x^{21} \\ x^{31} \end{bmatrix} \quad \begin{array}{l} \text{"ROCKET frame"} \\ \text{representation} \end{array}$$

- T.2 -
Transition from one basis ("frame") to another.

Consider the transformation from one frame to another. Relativity physics tells us that this transformation is typically

$$[\mathbf{x}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [\mathbf{x}]' \quad (3)$$

or

$$[\mathbf{x}]' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [\mathbf{x}] \quad (4)$$

In terms of index notation one has

$$x^\alpha = \Lambda^\alpha_{\alpha'} x^{\alpha'} \quad (4a)$$

or

$$x^{\alpha'} = \Lambda^{\alpha'}_\alpha x^\alpha \quad (4b)$$

Notice the consistent placement of the indices: in going from the old coordinate to the new coordinates, the corresponding labels of the transformation matrix Λ go from south-east to north-west "↖".

The matrices $\begin{bmatrix} \lambda^\alpha & \alpha_1 \\ \alpha_1 & \alpha \end{bmatrix}$ and $\begin{bmatrix} \lambda^{\alpha'} & \alpha' \\ \alpha' & \alpha \end{bmatrix}$ are inverses of each other:

$$\lambda^\alpha_{\alpha'} \lambda^{\alpha'}_\beta = \delta_\beta^\alpha \quad \text{Index notation}$$

$$\lambda^{-1} \lambda = I \quad \text{Matrix notation}$$

Consequently, the vector x has with the help of Eqs (1) and (4b) the expansion

$$\begin{aligned} x &= e_\alpha x^\alpha \\ &= e_\alpha \lambda^\alpha_{\alpha'} \lambda^{\alpha'}_\beta x^\beta \\ &= e_\alpha \lambda^\alpha_{\alpha'} x^{\alpha'} = e_{\alpha'} x^{\alpha'} \end{aligned}$$

This expansion holds for all vectors x , i.e. for all 4-tuples $\{x^{\alpha'}\}$. Consequently

$$e_\alpha \lambda^\alpha_{\alpha'} = e_{\alpha'}.$$

In summary, it is fruitful to combine the index and the matrix notation:

$$\begin{aligned} x &= [e_0, e_1, e_2, e_3] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = e_\alpha x^\alpha = e_{\alpha'} x^{\alpha'} = [e_0, e_1, e_2, e_3] \begin{bmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{bmatrix} \\ &= [e]^\alpha [x] = [e']^{\alpha'} [x'] \end{aligned}$$

-T,3-

-T,4-

$$x = e_{\alpha'} \lambda^{\alpha'}_{\alpha} x^\alpha = [e']^\alpha [x]$$

$$x = e_\alpha \lambda^\alpha_{\alpha'} \lambda^{\alpha'}_\beta x^\beta = [e]^\alpha \lambda^{-1} \lambda [x]$$

$$x = e_\alpha \lambda^\alpha_{\alpha'} x^{\alpha'} = [e]^\alpha \lambda^{-1} [x']$$

Note that

$$x = e_{\alpha'} \lambda^{\alpha'}_{\alpha} x^\alpha$$

can be read in two ways:

(i) x has components $\lambda^{\alpha'}_{\alpha} x^\alpha$ relative to the $e_{\alpha'}$ -basis.

OR
(ii) x has components x^α relative to the $e_{\alpha'} \lambda^{\alpha'}_{\alpha}$ -basis.

Note: $[x]'$ and $[x]$ refer to the same vector

-T. 5-

* , that is why the prime on
 $[x]'$ is on the outside: it labels the
coordinatization of spacetime.

By contrast $[e']^T$ and $[e]$ are
actually different sets of basis vectors.

-T. 6-

Advantages of index and matrix
notation:

Index notation: explicit reference to
the vector components relative to
the chosen basis.

Matrix notation: ease of transforming
from one basis representation to
another.

TUTORIAL II. / Index and Matrix Representation of Transformation. -T. 7-

Consider an active Lorentz transformation, which actually moves a 4-vector from, say, the LAB frame to the ROCKET frame. Under such a circumstance, the 4-vector itself changes so that one now has two different vectors, for example the ^{given} four-velocity at time t vs the ^{boosted} four-velocity at $t + \Delta t$.

$$\overset{+/-}{\rightarrow} \overset{T(S)}{\rightarrow} S = e_\alpha S^\alpha = [e] T[S] \xrightarrow{T} T(S) = e_\alpha T^\alpha, S^\beta = [e]^T T[S]$$

Thus at time t the LAB components of S are S^α , while at time $t + \Delta t$ we have a boosted, and hence different, vector $T(S)$ whose LAB components are T^α, S^β .

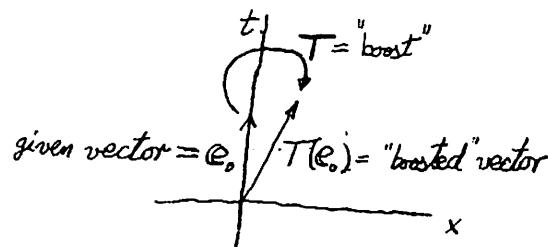
Example.

As an example consider the transformation -T. 8-

$$T = \begin{bmatrix} \text{ch}\theta & \text{sh}\theta & 0 & 0 \\ \text{sh}\theta & \text{ch}\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} T^0_0 & T^0_1 & \dots \\ T^1_0 & T^1_1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Its effect on LAB clock is to boost its velocity parameter by the positive amount θ so that it becomes a ROCKET clock:

$$T(e_0) = e_\alpha T^\alpha_0 = e_0 \text{ch}\theta + e_1 \text{sh}\theta = e_0$$



Problem:

-T, 9'

The previous transformation T acts on each of the four basis vectors e_0, e_1, e_2, e_3 and thereby produces the four corresponding new basis vectors

$$T(e_0) \equiv e_{0'}, T(e_1) \equiv e_{1'}, T(e_2) \equiv e_{2'}, T(e_3) \equiv e_{3'},$$

or more briefly

$$T(e_\alpha) = e_\beta T^\beta{}_\alpha \equiv e_{\alpha'} \quad (*)$$

Find the representation of the vector

$$S = e_\alpha S^\alpha = [e]^T [S] \quad (**)$$

relative to this new basis.

Solution:

The problem is to find the matrix Λ such that

$$S = e_{\alpha'} \Lambda^{\alpha'}{}_\alpha S^\alpha = [e']^T \Lambda [S]$$

Using Eq. (*), one obtains

$$S = e_\beta T^\beta{}_\alpha' \Lambda^{\alpha'}{}_\alpha S^\alpha = [e]^T T \Lambda [S]$$

Compare this with Eq. (**), use the fact that this holds for all 4-tuples [S], and conclude that

$$T^\beta{}_\alpha' \Lambda^{\alpha'}{}_\alpha = \delta^\beta{}_\alpha \text{ OR } T\Lambda = I \text{ (= identity matrix)}$$

Consequently,

$$I = T^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T, 10

Conclusion: An active transformation such as T has matrix elements which form the inverse of ^{the} transition matrix Λ which acts on the old coordinates S^α of S to produce the new coordinates S'^α of S

$$[S']^T = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [S]$$

↑
new
coordinates
of the same
vector S

↑
old
coordinates
of the same
vector S

$\Lambda (= T^{-1})$

On the other hand

$$[e_0, e_1, e_2, e_3] = [e_0, e_1, e_2, e_3] \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

new "boosted" basis old basis

T

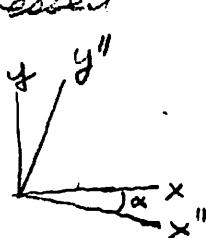
TUTORIAL
III. Change of Representation.

-T.11-

Introduce new axes which are related to the old (x, y) -axes by the angle α clockwise.

This change of frame is expressed by the matrix

$$R(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The vector at time t is now represented in two ways, namely

$$S = e_\mu S'' = [e]^\top [S] = [e]^\top R R^{-1} [S]$$

$$S = [e'']^\top [S'']$$

$$= e_{\mu''} S''$$

$$\text{where } e_{\mu''} = e_\mu R''(\alpha)$$

$$[e''] = [e] R$$

$$S'' = R''(\alpha) S'$$

$$[S''] = R^{-1} [S]$$

Similarly at time $t+t$ the transformed vector is also represented in two ways

$$T(S) = e_\mu T'', S'' = [e]^\top T [S]$$

$$= [e] R R^{-1} T R R^{-1} [S]$$

$$= [e''] R'' T R [S'']$$

and

$$T(S) = e_{\mu''} (R^{-1} T R)'' S''$$

$$\text{where } (R^{-1} T R)'' = R'' \alpha T'' \beta R'' \gamma'' .$$

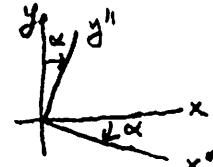
The matrix $R^{-1} T R$ is called the representation of T relative to the rotated coordinate frame $[e_0'' e_1'' e_2'' e_3'']$.

Problem:

Given: A boost transformation into the positive x -direction,

$$T = \begin{bmatrix} 1 & \gamma & 0 & 0 \\ \rho\gamma & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Find its matrix elements relative to the coordinate axes (x'', y'') rotated clockwise by α



-T.13-

Solution:

$$\begin{aligned}
 R^{-1} T R &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma \cos\alpha & \gamma \cos\alpha & -\sin\alpha & 0 \\ 0 & \gamma \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \downarrow \\
 &= \begin{bmatrix} \gamma & \beta\gamma \cos\alpha & \beta\gamma \sin\alpha & 0 \\ \beta\gamma \cos\alpha & \gamma \cos^2\alpha + \sin^2\alpha & (\gamma-1) \sin\alpha \cos\alpha & 0 \\ 0 & (\gamma-1) \sin\alpha \cos\alpha & \gamma \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \gamma & \beta\gamma \cos\alpha & \beta\gamma \sin\alpha & 0 \\ \beta\gamma \cos\alpha & 1 + (\gamma-1)\cos^2\alpha & (\gamma-1) \sin\alpha \cos\alpha & 0 \\ 0 & (\gamma-1) \sin\alpha \cos\alpha & 1 + (\gamma-1)\sin^2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

This is the transformation matrix which represents T relative to the rotated basis $[e_0'', e_1'', e_2'', e_3'']$

$$e_{\mu''} \rightarrow e_{\nu''} (RTR^{-1})^{\nu'' \mu''} \quad (1)$$

$$[e'']^T \rightarrow [e'']^T RTR^{-1} = [e'']^T \quad (2)$$

↑ original basis vectors ↓ new basis vectors

-T.14-

Problem:

Suppose the components of vector S

are S'' , i.e.

$$S = [e''] S'' = [e'']^T [S]'' \quad (3)$$

Find its components relative to the new basis $[e'']^T RTR^{-1}$, i.e. find the Lorentz transformation matrix Λ which expresses the new components in terms of the old components.

$$S = [e'']^T [S]'''$$

$$= [e'']^T \Lambda [S]'''$$

Using Eq. (1) one obtains

$$S = [e'']^T RTR^{-1} \Lambda [S]'''$$

This expression is equal to Eq. (3) for every vector S , i.e. for all possible $[S]''$. Consequently,

$$RTR^{-1} \Lambda = I$$

or

$$\boxed{\Lambda = RTR^{-1}}$$

$$RTR^{-1} [S]''' = [S]''$$

Conclusion: The "active" transformation and the corresponding coordinate transformation are inverses of each other.

③ Thomas Precession:
Quantitative Determination.

There are two comoving inertial frames, namely A and B, and both are related to the LAB frame by the pure boost transformations $\Lambda(t)$ and $\Lambda(t+\alpha t)$ of A and B respectively.

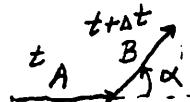
$\Lambda(t+\alpha t)$ differs from $\Lambda(t)$ in that the boost velocity of $\Lambda(t+\alpha t)$ is at an angle α relative to that of $\Lambda(t)$

$$A \text{ at time } t : [\mathbf{x}]_A = \Lambda(t) [\mathbf{x}]_L = \begin{bmatrix} 1 & -\beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [\mathbf{x}]_L$$

COMOVING coordinates of A relative to { B

$$B \text{ at time } t+\alpha t : [\mathbf{x}]_B = \Lambda(t+\alpha t) [\mathbf{x}]_L = R(-\alpha) \Lambda(t) R(\alpha) [\mathbf{x}]_L$$

We have $R(-\alpha)$ because α for here in this case



is counterclockwise whereas on P.T. II it was clockwise.

a) Here

$$R(-\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= I - \alpha L_3, \quad \alpha \ll 1$$

is the transition matrix which rotates the x and y coordinates by the angle α counter clockwise.

-2-

- 3 -

b) Consequently,

$$\Lambda(t+\alpha) = R(-\alpha) \Lambda(t) R^{-1}(\alpha)$$

is a pure coordinate boost along the direction α .

Thus we have two pure boosts

$$\Lambda(t) \text{ and } \Lambda(t+\alpha) = R(-\alpha) \Lambda(t) R^{-1}(-\alpha),$$

But because they are boosts into different directions,
the composite

$$\Lambda(t+\alpha) \Lambda(t) = R(-\alpha) \Lambda(t) R^{-1}(-\alpha) \Lambda^{-1}(t)$$

is not a pure boost; it is a boost together with
a spatial rotation. This means that if

(i) we take a spacetime displacement x
(i.e. a 4-vector) whose coordinates
are $[x]_A$ relative to comoving inertial
frame A, (ii) transform its components to
the LAB using $\Lambda'(t)$

$$[x]_L = \Lambda'(t) [x]_A,$$

and finally

- 4 -

(iii) transform these back to comoving
inertial frame B using $\Lambda(t+\alpha)$

$$[x]_B = \Lambda(t+\alpha) [x]_L$$

$$[x]_B = R(-\alpha) \Lambda(t) R^{-1}(-\alpha) \Lambda'(t) [x]_A$$

one finds that $[x]_B$ is related to $[x]_A$
by a pure boost and a spatial rotation.

Let us calculate what that
rotation and what that boost is.

c) The rotation and boost calculation
is readily done. One needs this only for $\alpha \ll 1$,
thus one has

$$\begin{aligned} R(-\alpha) \Lambda(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma \beta & 0 & 0 \\ \gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \left[I + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \downarrow \\ & & & \Lambda(t) \end{bmatrix} \end{aligned}$$

$$= \Lambda(t) + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\gamma \beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

Similarly, by replacing $\alpha \rightarrow -\alpha$ and $\beta \rightarrow -\beta$, one obtains the product of the inverse matrices

-5-

$$R^{-1}(-\alpha) \tilde{A}^{-1}(t) = \tilde{A}^{-1}(t) - \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so that finally

$$A(t+\alpha t) \tilde{A}^{-1}(t) = \underbrace{R(-\alpha) A(t)}_{(*)} \underbrace{R^{-1}(-\alpha) \tilde{A}^{-1}(t)}$$

$$= \left[\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= I + \alpha \begin{pmatrix} 0 & 0 & -\gamma\beta & 0 \\ 0 & 0 & \gamma-1 & 0 \\ -\gamma\beta & 1-\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \text{negligible term because } \alpha \ll 1$$

$$= I - \alpha \gamma \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha(\gamma-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \dots$$

$$= \left[I + \alpha(\gamma-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \dots \right] \left[I - \alpha \gamma \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \dots \right] \quad (**)$$

spatial rotation
in the (xy) -plane
counter clockwise
by an angle α .

pure boost
along the y -direction

$$\equiv [I + \alpha(\gamma-1)L_3 + \dots][I - \alpha \gamma \beta K_2]$$

-6-

This transformation says that the spacetime displacement x viewed relative to the comoving frame B is measured to have been spatially rotated by an amount $\alpha(\gamma-1)$ counterclockwise, and is measured to have been boosted by an amount $-\alpha \gamma \beta$ relative to the measurement of x relative to the comoving frame A .

Algebraically one expresses this conclusion by the statement

$$A(t+\alpha t) \tilde{A}^{-1}(t) = [I + \alpha(\gamma-1)L_3 + \dots][I - \alpha \gamma \beta K_2 + \dots] = [I - \alpha \gamma \beta K_2][I + \alpha(\gamma-1)L_3] + \text{negligible terms}$$

where

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are called "rotation and boost generators".

The angle α is very small. Consequently one gets the same answer if one does the spatial rotation first, and then does the pure boost, i.e. for $\alpha \ll 1$ the boost and the rotation commute:

$$\Lambda(t+\alpha t)\bar{\Lambda}(t) = \underbrace{[I - \alpha \gamma \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}]}_{\text{pure boost}} \underbrace{[I + \alpha(\gamma^{-1}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}]}_{\text{spatial rotation}}$$

The problem with the transformation

$$[\mathbf{x}]_B = \Lambda(t+\alpha t)\bar{\Lambda}(t)[\mathbf{x}]_A$$

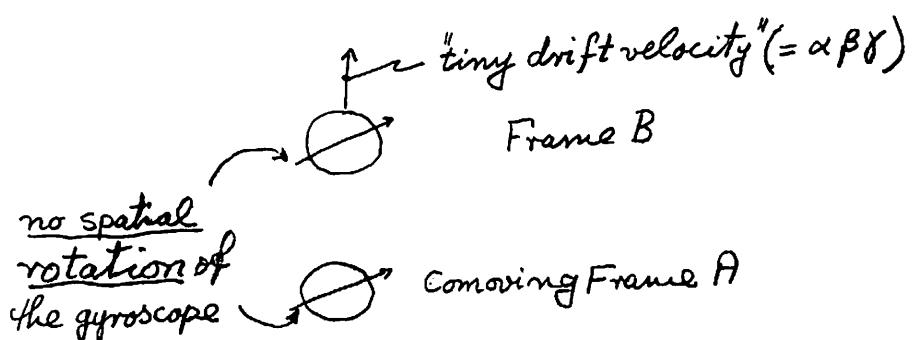
is that it is merely a change in the coordinates

$$[\mathbf{x}]_A = \begin{bmatrix} x_A^0 \\ x_A^1 \\ x_A^2 \\ x_A^3 \end{bmatrix} \text{ to } [\mathbf{x}]_B = \begin{bmatrix} x_B^0 \\ x_B^1 \\ x_B^2 \\ x_B^3 \end{bmatrix}$$

of one and the same fixed vector \mathbf{x} . This change refers to the change from the

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basis vectors comoving with frame A to the basis vectors comoving with frame B. What is needed is how a moving vector, e.g. the spin carried by an accelerated electron or gyroscope, changes as it evolves in moving from comoving frame A to comoving frame B. Observations reveal that the gyroscope undergoes no spatial rotations as it transits from comoving frame A to comoving frame B.



With this observation in mind one now considers two vectors $\mathbf{x} = \mathbf{S}(t)$ and $\mathbf{S}(t+\Delta t)$.

The second vector $\mathbf{S}(t+\Delta t)$ differs from $\mathbf{S}(t)$ in that the components of $\mathbf{S}(t+\Delta t)$ measured relative to comoving frame B do not exhibit any rotation as compared to the measured components of $\mathbf{x} = \mathbf{S}(t)$. This means that

$$\mathbf{S}(t+\Delta t) = [\mathbf{e}^B]^T \left[\mathbf{S}(t+\Delta t) \right]_B = [\mathbf{e}^B]^T \left[I - \alpha \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[\mathbf{S}(t) \right]_A$$

while

$$\begin{aligned} \mathbf{S}(t) \equiv \mathbf{x} \equiv [\mathbf{e}^B]^T \left[\mathbf{S}(t) \right]_B &= [\mathbf{e}^B]^T \underbrace{\left[\Lambda(t+\Delta t) \Lambda^{-1}(t) \right]}_{\text{boost and rotation}} \left[\mathbf{S}(t) \right]_A \\ &= [\mathbf{e}^B]^T \left[I + \alpha(\gamma-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[I - \alpha \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[\mathbf{S}(t) \right]_A \end{aligned}$$

as one can readily see by referring to Eqs. (**) and (*) on P 5,

We finally ask the question posed by the LAB observer, namely what are the components of the two vectors measured relative to the LAB frame?

Recalling from P 3 and 4 the relation between the components measured in the LAB and the COMOVING frames,

$$\left[\mathbf{S}(t) \right]_A = \Lambda(t) \left[\mathbf{S}(t) \right]_L$$

$$\left[\mathbf{S}(t+\Delta t) \right]_B = \Lambda(t+\Delta t) \left[\mathbf{S}(t+\Delta t) \right]_L$$

$$[\mathbf{e}^B]^T = [\mathbf{e}^L]^T \Lambda^{-1}(t+\Delta t)$$

one finds

$$\mathbf{S}(t+\Delta t) = [\mathbf{e}^L]^T \Lambda^{-1}(t+\Delta t) \left[I - \alpha \beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \Lambda(t) \left[\mathbf{S}(t) \right]_L$$

versus

$$\mathbf{S}(t) = [\mathbf{e}^L]^T \left[\mathbf{S}(t) \right]_L$$

as expected.

All components of $S(t+\Delta t)$ measured relative to the LAB frame can be read out from the matrix

$$M = \bar{\Lambda}^{-1}(t+\Delta t) \left[I - \alpha\beta\gamma \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{K_2} \right] \Lambda(t)$$

$$= \underbrace{\bar{\Lambda}^{-1}(t+\Delta t)\Lambda(t)}_{1st \text{ term}} - \alpha\beta\gamma \underbrace{\bar{\Lambda}^{-1}(t+\Delta t)K_2\Lambda(t)}_{2nd \text{ term}}$$

1) Calculate the 1st term. From P 5 & 6 one has to first order in $\alpha \ll 1$:

$$\Lambda(t+\Delta t)\Lambda(t) \equiv R(-\alpha)\Lambda(t)R^{-1}(-\alpha)\bar{\Lambda}^{-1}(t)$$

$$\approx I + \alpha(\gamma-1)L_3 - \alpha\beta\gamma K_3 + \dots$$

where

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } K_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\boxed{\bar{\Lambda}^{-1}(t+\Delta t)\Lambda(t) = R(-\alpha)\bar{\Lambda}^{-1}(t)R^{-1}(-\alpha)\Lambda(t)}$$

$$= \text{same as above, but } \beta \rightarrow -\beta$$

$$= \boxed{I + \alpha(\gamma-1)L_3 + \alpha\beta\gamma K_3 + \dots} \quad (*)$$

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2) Calculate the 2nd term. To zero order in $\alpha \ll 1$ one has from P 1

$$\bar{\Lambda}^{-1}(t+\Delta t) \approx \bar{\Lambda}^{-1}(t) = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Consequently

$$\bar{\Lambda}^{-1}K_2\Lambda = \bar{\Lambda}^{-1} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Lambda = K_2 \quad (***)$$

Apply Eqs (**) and (***) to M and one has

$$M = \bar{\Lambda}^{-1}(t+\Delta t)\Lambda(t) - \alpha\beta\gamma \bar{\Lambda}^{-1}K_2\Lambda$$

$$= I + \alpha(\gamma-1)L_3 + \alpha\beta\gamma K_3 - \alpha\beta\gamma K_2$$

$$= I + \alpha(\gamma-1)L_3$$

Thus we see that as measured in $LAB^{-1}3-$
the initial four-vector $S(t)$ at A and
the subsequent four-vector $S(t+\Delta t)$ at B are
related by

$$\begin{aligned} S(t+\Delta t) &= [e^L]^T M [S(t)]_L \\ &= [e^L] \left[I + \alpha(\gamma-1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] [S(t)]_L \end{aligned}$$

$$\text{where } \alpha = \frac{2\pi}{n}$$

Let Ω_2 = angular velocity of the orbiting gyroscope.

Let $T = \frac{2\pi}{\Omega_2}$ = orbital period.

$$\text{Then } \Delta t = T \frac{\alpha}{2\pi}$$

$$\text{or } T = n \Delta t$$

Consequently

$$\begin{aligned} S(t+T) &= [e^L]^T M^n [S(t)]_L \\ &= [e^L]^T \left[I + \frac{2\pi(\gamma-1)}{n} L_2 \right]^n [S(t)]_L \end{aligned}$$

$$\begin{aligned} S(t+T) &= [e^L]^T e^{\varphi L_2} [S(t)]_L \quad -14- \\ &= [e^L]^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi & 0 \\ 0 & -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [S(t)]_L \end{aligned}$$

$$\text{where } \varphi = 2\pi(\gamma-1)$$

is the clockwise precession

angle of the gyroscope
for each orbital period. The angular
precession frequency is

$$\omega = \frac{\varphi}{T} = \frac{2\pi(\gamma-1)}{\frac{2\pi}{\Omega_2}} = \Omega_2(\gamma-1) = \begin{pmatrix} \text{orbiting} \\ \text{angular} \\ \text{frequency} \end{pmatrix} (\gamma-1).$$

$$\boxed{\omega = \Omega_2(\gamma-1)} \quad \text{"clockwise"}$$

This is the Thomas precession frequency
whose sense agrees with the qualitative
discussion based on the relativity of
simultaneity on P. 5 on the
handout "Thomas precession: Its essentials"