

## Lecture 10

One parameter family of  
Instantaneous Lorentz frames  
("moving")

along a given world line:

Fermi-Walker transport  
[MTW Ch 6]

## Fermi-Walker Transport

Given a generic worldline

$$X(\tau) = \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\}$$

we would like to construct along it a moving frame which from its own perspective is nonrotating ("no precession of coaccelerating gyroscopes").

Such a moving frame consists of a one parameter family of Lorentz orthonormal basis vectors. At some initial event, say,

$X(\tau) = P$  one constructs the frame

$$e_0(\tau) = u = \frac{dX}{d\tau} = \text{tangent to worldline } X(\tau),$$

$$e_1(\tau) = \frac{1}{g} \frac{d^2 X}{d\tau^2} = \frac{a}{g} = \frac{1}{g} \frac{du}{d\tau} = \text{"1st normal" to worldline } X(\tau);$$

$$a \cdot a = g^2$$

$e_2(\tau)$  } two space-like  
 $e_3(\tau)$  } unit vectors  
 normal to  $u$  and  $a$ .

-8-

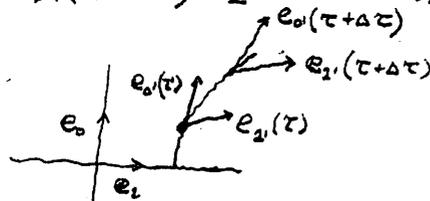
Reminder: The primes on the subscripts -9-

designate that one has a comoving and  $\tau$ -dependent basis, which is to be distinguished from the basis

$$\{e_0, e_1, e_2, e_3\}$$

which is fixed relative to the fixed stars, and hence does not depend on  $\tau$ .

Having constructed a Lorentz orthonormal frame at  $P = X(\tau)$ , one then constructs at  $X(\tau + \Delta\tau) = P + \Delta P$  that unique frame which



- (1) is related to that at  $P$  by a Lorentz transformation
- (2) has worldline-tangent  $e_0(\tau + \Delta\tau) = \left. \frac{dX}{d\tau} \right|_{\tau + \Delta\tau}$  as its vector  $e_0(\tau + \Delta\tau)$ , i.e.,  
 $e_0(\tau) = u(\tau) \quad \forall \tau$ .
- (3) is a "pure boost", i.e., no spatial rotation relative to freely gimbaled gyroscopes.

A frame moving along the worldline by these three rules is called a Fermi-Walker transported frame. The three spacelike basis vectors can be viewed as being determined by three orthogonal gyroscopes. These basis vectors together with the four-velocity of the worldline make up a F-W transported (and hence Lorentz) orthonormal tetrad. This rigid tetrad undergoes a rotational <sup>(relative to the fixed stars)</sup> motion which is determined entirely by the curvature properties of the worldline.

A four-vector in the moving frame, i.e. a one parameter family of vectors located on the worldline, is said to undergo F-W transport if its coordinate components relative to the moving frame do not change.

We shall see that the three conditions at the bottom of P 9, namely,

- (1) vectors are related by Lorentz transformation
- (2) unit tangent is one of the basis vectors
- (3) only "pure boost", i.e. no rotations

[Read also §6.5 in MTW]

determine a unique  $\tau$ -parametrized frame (in mathematics, a "basis")

$$\{e_0(\tau), e_1(\tau), e_2(\tau), e_3(\tau) : -\infty < \tau < \infty\}$$

along the given worldline  $X(\tau)$ .

Definition:

One says that a moving vector  $V(\tau)$ ,  $0 < \tau < \infty$ , is Fermi-Walker transported along  $X(\tau)$  if its components remain fixed relative to the above uniquely determined

basis:

a)  $V(\tau) = v^{\mu'} \underbrace{e_{\mu'}}_{\text{fixed}}(\tau)$

In this representation of  $V(\tau)$  the basis  $\{e_{\mu'}(\tau)\}$  changes relative to the fixed stars, but the components  $v^{\mu'}$  remain fixed.

However, a more relevant basis is the LAB basis (on P 9) which is fixed relative to the fixed stars. Relative to it the vector  $V(\tau)$ , Eq. (\*), is represented

b) by  $V(\tau) = v^{\mu}(\tau) \underbrace{e_{\mu}}_{\text{fixed relative to stars}}(\tau)$  (\*\*)

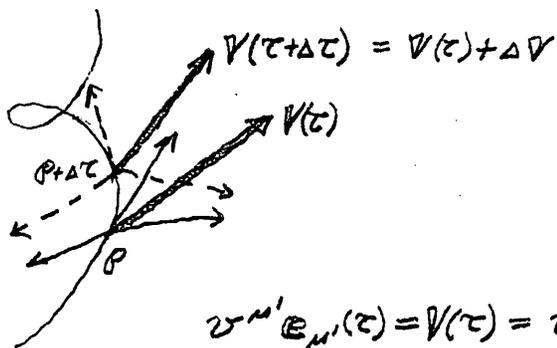
with  $\tau$ -dependent coefficients. (Nota bene: the lack of any primes on the summation index  $\mu$  implies that we are using a fixed LAB basis.)

c) Question: What are the coefficients  $v^{\mu}(\tau)$  which guarantee that

$v^{\mu}(\tau) e_{\mu} = V(\tau) = v^{\mu'} e_{\mu'}(\tau)$

for all  $\tau$ ?

Answer: We shall determine these coefficients by setting up the differential equation they satisfy.



$v^{M'} e_{M'}(\tau) = V(\tau) = v^M(\tau) e_M$   
 ↑  
 Comoving components (fixed relative to gyroscope determined basis)      LAB basis vector (fixed relative to fixed stars)

$V(\tau + \Delta\tau)$  and  $V(\tau)$  are related by a Lorentz transformation  $\Lambda^M{}_{\nu}$ ; hence

at  $P$ :  $v^M(\tau)$

at  $P + \Delta P$ :  $v^M(\tau + \Delta\tau) = \Lambda^M{}_{\nu}(\tau + \Delta\tau) v^{\nu}(\tau)$

" =  $\left[ \Lambda^M{}_{\nu}(\tau) + \Delta\tau \Omega^M{}_{\nu} + \frac{(\Delta\tau)^2}{2!} (\dots)^M{}_{\nu} \right] v^{\nu}(\tau)$

$v^M(\tau) + \Delta v^M = \left[ \delta^M{}_{\nu} + \Delta\tau \Omega^M{}_{\nu} + \dots \right] v^{\nu}(\tau)$   
 Thus the coefficients  $\{\Omega^M{}_{\nu}\}$  satisfy the following system of QDE's,

$\boxed{\frac{dv^M}{d\tau} = \Omega^M{}_{\nu} v^{\nu}} \quad M = 0, 1, 2, 3$

The matrix  $\Omega^M{}_{\nu}$  is called the generator of the Lorentz transformation,

Such a generator has a simple property which becomes evident if we introduce

the matrix  $[\eta_{\sigma\rho}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

and its inverse

$[\eta^{\nu\sigma}] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

so that

$[\eta^{\nu\sigma} \eta_{\sigma\rho}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv [\delta^{\nu}{}_{\rho}]$  "Kronecker delta"

In terms of these one has

$$\begin{aligned} \Omega^M{}_{\nu} v^{\nu} &= \Omega^M{}_{\nu} \delta^{\nu}{}_{\rho} v^{\rho} \\ &= \underbrace{\Omega^M{}_{\nu}}_{\Omega^{M\sigma}} \underbrace{\eta^{\nu\sigma} \eta_{\sigma\rho}}_{v^{\rho}} v^{\rho} \\ &= \Omega^{M\sigma} v^{\rho} \end{aligned}$$

Thus the differential equation for the components of the vector  $V$  are

$$\frac{dv^M}{d\tau} = \Omega^M{}_\nu v^\nu \quad \frac{dv^M}{d\tau} = \Omega^M{}_\nu v^\nu$$

$\Omega^{\mu\nu}$  is determined by the worldline  $P = X(\tau)$  as follows:

2. The length preserving Lorentz rotation implies

$$v^\mu v^\nu \eta_{\mu\nu} = (v^\mu + \Delta v^\mu)(v^\nu + \Delta v^\nu) \eta_{\mu\nu}$$

or to first order

$$0 = (v^\mu \Delta v^\nu + \Delta v^\mu v^\nu) \eta_{\mu\nu} = \Delta\tau (v_\nu \Omega^{\nu\sigma} v_\sigma + \Omega^{\mu\sigma} v_\sigma v_\mu)$$

$$= \Delta\tau v_\nu (\Omega^{\nu\sigma} + \Omega^{\sigma\nu}) v_\sigma \quad \begin{matrix} \mu = \sigma \\ \sigma = \nu \end{matrix} \text{ \& drop the " - "}$$

with the help of the differential equation and a relabelling of indices.

The length of every vector is to be unchanged.

This is possible only if

$$\Omega^{\nu\sigma} = -\Omega^{\sigma\nu} \quad (1)$$

In other words, the "generator"  $\Omega^{\nu\sigma}$  of the Lorentz transformation

$$\Lambda^M{}_\nu(\tau+\Delta\tau) = (\delta^M{}_\nu + \Omega^M{}_\nu \Delta\tau + \dots)$$

is to be antisymmetric.

3. The individual elements of the antisymmetric  $\Omega^{\mu\nu}$  are determined by the requirement that a Fermi-Walker transported vector undergoes a change which is a pure boost in the  $u$ - $z$  plane, meaning no rotation relative to the accelerated but freely gimbaled gyroscopes. The change for any such vector  $v^M$  is therefore

$$\Omega^{\mu\nu} v_\nu = \frac{dv^M}{d\tau} = a u^M + b a^M$$

This determines the entries of the antisymmetric  $\Omega^{\mu\nu}$ . Indeed, let

$$u \equiv \frac{dX}{d\tau} : \left\{ \frac{dx^\mu(\tau)}{d\tau} \equiv u^\mu(\tau); \mu=0,1,2,3 \right\}$$

$$a \equiv \frac{d^2X}{d\tau^2} : \left\{ \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} \equiv a^\mu(\tau); \mu=0,1,2,3 \right\}$$

with

$$a \cdot a = a^\mu(\tau) a^\nu(\tau) \eta_{\mu\nu} \equiv g^2(\tau)$$

$$W_1 : \left\{ w_1^\mu(\tau); \mu=0,1,2,3 \right\}$$

$$W_2 : \left\{ w_2^\mu(\tau); \mu=0,1,2,3 \right\}$$

be a F-W transported basis. Then  $\Omega^{\mu\nu}$  in fact any antisymmetric matrix, necessarily has the form

$$\begin{aligned} \Omega^{\mu\nu} = & \alpha (u^\mu a^\nu - a^\mu u^\nu) \\ & + \beta (u^\mu w_1^\nu - w_1^\mu u^\nu) \\ & + \gamma (u^\mu w_2^\nu - w_2^\mu u^\nu) \\ & + \delta (a^\mu w_1^\nu - w_1^\mu a^\nu) \\ & + \epsilon (a^\mu w_2^\nu - w_2^\mu a^\nu) \\ & + \theta (w_1^\mu w_2^\nu - w_2^\mu w_1^\nu), \end{aligned}$$

where the constants  $\alpha, \beta, \gamma, \delta, \epsilon, \theta$  are uniquely determined as follows:

Applying the transport differential equation

$$\Omega^{\mu\nu} v_\nu = a u^\mu + b a^\mu$$

to  $v_\nu = a_\nu, u_\nu,$  and  $w_{1\nu}$

one finds that their linear independence in relation to  $w_{2\nu}$  yields

$$\delta = \epsilon = 0, \quad \beta = \gamma = 0, \quad \text{and } \theta = 0, \text{ respectively}$$

The fact that  $a^\nu u_\nu = 0$  and  $u^\nu u_\nu = -1$

yields

$$\begin{aligned} a^\mu &= \frac{du^\mu}{d\tau} = \Omega^{\mu\nu} u_\nu = \alpha (u^\mu a^\nu - a^\mu u^\nu) u_\nu \\ &= \alpha a^\mu. \end{aligned}$$

Consequently

$$\boxed{\frac{dv^\mu}{d\tau} = (u^\mu a^\nu - a^\mu u^\nu) v_\nu} \quad \mu=0,1,2,3$$

$= \eta_{\nu\sigma} v^\sigma$

are the differential eq'ns for a F-W transported vector

$$V = v^\mu(\tau) E_\mu.$$

6 amper to 20800 ampere  
on p 90 of Lecture 9

Comment: The boxed

equation can be summarized by saying that the effect of F-W transport consists of pure boost confined to the longitudinal direction; in other words, the F-W subjects the frame  $(\{u^\mu(\tau)\} \{a^\mu(\tau)\} \{w_1^\mu(\tau)\} \{w_2^\mu(\tau)\})$  to a motion which consists of pure boost along  $u$  and  $a$  without any spatial rotation, which would impart to a vector a change perpendicular to the  $u-a$  plane.

See the spiral Thomas precession.

Comment: This lack of "spatial rotation" prevents a F-W transported vector from acquiring any components transverse to the  $u-a$  plane. This does not mean that a F-W transported vector can not rotate spatially relative to an inertial frame. It can, but that happens only if the worldline makes the  $u-a$  plane rotate, and

[End of lecture 10]

to the and that's what gives rise to the