

# Lecture 14

## Metric (cont'd)

Natural isomorphism betw.  $V$  &  $V^*$

Basis representation of  $g$  and  
its use in transforming vectors  
into covectors.

The Reciprocal Basis for  $V$

$V \xrightarrow{g} V^*$   
 "Natural" isomorphism

14.1  
~~14.1~~

Comment: A metric establishes a natural isomorphism between the vector space  $V$  and its space of duals,  $V^*$ . In order to conserve notation we shall use the same symbol  $g$  to designate this correspondence. Its defining property is

A natural isomorphism is independent of any chosen basis.

Basis independent definition

$$g: V \rightarrow V^*$$

$x \rightsquigarrow g(x, \cdot) = \sum_{\text{linear fn}} x_i (\text{"}x\text{"})$

Here  $x$  is that linear functional which, when operating on  $y \in V$ , yields  $g(x, y)$ :

$$x = "x": V \xrightarrow{x} R$$

$$y \rightsquigarrow \langle x, y \rangle = x \cdot y = g(x, y)$$

One can use the representation of the metric relative to a given basis present

$$g = g_{ij} \omega^i \otimes \omega^j$$

to implement the isomorphism  $V \rightarrow V^*$  on the components of the vectors in  $V$  and the corresponding covectors in  $V^*$ . This is done as follows:

Basis rep'n of  $g$

14.2  
~~14.2~~

Proposition: (Basis representation of  $g$ ).

Let  $\{e_i\}$  be a basis of  $V$   
 Let  $\{\omega^j\}$  be its dual basis for  $V^*$

Then  $g$  can be written in terms of a "tensor product basis" as follows:

Basis dependent definition

$$g = g_{ij} \omega^i \otimes \omega^j: V \rightarrow V^*$$

$$x \rightsquigarrow g(x, \cdot) = g_{ij} \langle \omega^i | x \rangle \omega^j$$

$$e_i \rightsquigarrow g(e_i, \cdot) = g_{ij} \omega^j$$

Comment:

1. Here the tensor product sign  $\otimes$  establishes an ordered juxtaposition of pairs of linear functionals. This is the means by which a bilinear functional is constructed from a pair of linear functionals. The tensor product is therefore a means of obtaining a bilinear functional from two linear functionals.

Component

2. The components of  $g$  are obtained by evaluating it on pairs of vectors

$$g(e_k, e_l) = g_{ij} \langle \omega^i | e_k \rangle \langle \omega^j | e_l \rangle$$

$$= g_{ij} \delta_k^i \delta_l^j$$

$$= g_{kl} (= g_{lk})$$

(next page)  
~~(over)~~

14.3 ~~12.2~~

More generally, evaluating  $g$  on the pair of vectors  $x, y \in V$ , one obtains their inner product

$$\begin{aligned}x \cdot y &= g(x^k e_k, y^l e_l) = g_{kl} x^k y^l \\ &= e_k \cdot e_l x^k y^l \\ &\equiv g_{kl} x^k y^l\end{aligned}$$

The inner product  $g = \cdot$  establishes an isomorphism between  $V$  and  $V^*$ .

Question: How is this done for individual vectors?

Answer: To express this isomorphic relationship in terms of specific numbers one must first introduce a specific basis, say  $\{e_k\}$  for  $V$  and its corresponding dual basis  $\{\omega^i\}$  for  $V^*$ .

$$\{\omega^i | e_k\} = \delta^i_k.$$

A calculation establishes the relation by means of the following

Proposition:

Given:  $x = x^k e_k \in V$

Conclusion: The numerical coefficients  $x_i$  of the corresponding  $x = x_i \omega^i \in V^*$  are given by the following

$x$   
in  $V$

computation:

$$\begin{aligned} x &= g(x, \cdot) = g_{ij} \omega^i \otimes \omega^j (x^k e_k) \\ &= g_{ij} \langle \omega^i | x^k e_k \rangle \omega^j \\ &= g_{ij} \delta^i_k x^k \omega^j = x^k g_{kj} \omega^j \\ &= x_j \omega^j \in V^* \end{aligned}$$

$\{x^k\} \mapsto$

$x_j = g_{jk} x^k$

components of  $x \in V^*$   
"LOWERING" THE INDEX.

## The Reciprocal Basis

12-4  
14.5

Without a metric  $g$  there is no such thing as the length of a vector, the angle between two vectors, two vectors being perpendicular, etc.

With a metric  $g$  we have all these concepts. Moreover, given a basis  $\{e_i\}$ , the metric  $g$  determines the reciprocal basis  $\{e_i^+\}$  corresponding to the given bases. Thus  $V$  now has two bases,  $\{e_i\}$  and  $\{e_i^+\}$ .

The reciprocal basis is useful because, among other things, it sharpens the relation of the dual elements in  $V^+$  to those in  $V$ .

14.6  
12-5

The reciprocal basis and its properties arise as follows.

Definition: (reciprocal basis)

Given  $(g = \cdot)$ , a metric on  $V$

(ii)  $\{e_i\}$ , a basis for  $V$

Then the set of vectors

$$\{e_1^+, e_2^+, \dots, e_n^+\},$$

where

$$e_k^+ \cdot e_i = \delta_{ki},$$

is the basis reciprocal to  $\{e_i\}$ .

Comment: 1.  $e_k^+$  is a vector perpendicular to the plane spanned by  $\{e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n\}$ ,  
i.e.,

$$e_k^+ \cdot e_i = 0 \quad i \neq k$$

2.

$e_k^+$  is scaled such that

$$e_k^+ \cdot e_k = 1 \quad \text{N.O.SUM}$$

3.  $\{e_i^+\}$  is reciprocal to  $\{e_i\}$  and vice versa.

Proof: ①  $e_i^* \cdot e_j = g(e_i^*, e_j) = \delta_{ij} = \delta_i^j$   
 ② Evaluate  $\sigma^k$  for any basis vector  $e_i$ :  
 $\sigma^k(e_i) = g(e_i^*, e_i) = \delta_{ik}$   
 ③ These are the properties of  $\delta_i^k$ . Hence  
 $e_i^* \cdot e_j = \delta_i^j (= e_i^* \cdot e_j)$

Both the reciprocal basis vectors and the dual 1-forms are useful because they serve as projection operators. In fact we have

Proposition: (Projection operators)

The dual basis as well as the reciprocal basis elements serve as projection operators that yield the coordinates of a vector:

$$\langle \omega^k | \vec{x} \rangle = \langle \omega^k | x^i \vec{e}_i \rangle = x^i \delta_{ik} = x^k$$

$$\vec{e}_k^* \cdot \vec{x} = x^i \vec{e}_k^* \cdot e_i = x^i \delta_{ki} = x^k$$

$x^k =$  is the parallel projection of  $\vec{x}$  onto the coordinate axis of  $e_k =$   
 $=$  contravariant component of  $\vec{x}$ .

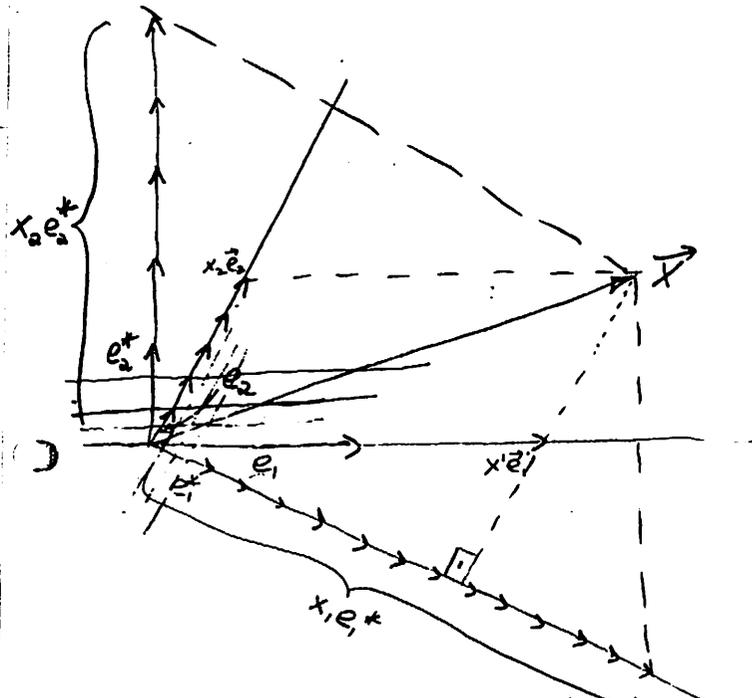
This differs from

$$e_j \cdot \vec{x} = e_j \cdot x^i \vec{e}_i = x^i g_{ji} = x_j = \text{"covariant" component}$$

~~12.7~~ ~~2111~~

In other words,

$$\vec{x} = x^i \vec{e}_i = \sum x_k^* e_k^* = x^1 e_1 + x^2 e_2 = x_1 e_1^* + x_2 e_2^*$$



In terms of these components the square length of a vector has the simple form.

$$x \cdot x = (x^1 e_1 + x^2 e_2) \cdot (x_1 e_1^* + x_2 e_2^*)$$

$$= x^1 x_1 + x^2 x_2$$

$$= x^1 g_{11} x^1 + x^2 g_{22} x^2$$

$$= g_{ii} x^i x^i$$