

Lecture 15

Vector normal to the isograms of a linear fr.
Bragg Diffraction

Tensors as multilinear maps;

Example, Definition and more examples

for [Tensor product
Tensor product basis.

Lecture 16 [Tensor space

New tensors via (i) lowering indices and
(ii) contraction

[Read §3.2 in MTW]

3.5

4.1, 4.2

14.9 21.12

The fact that elements of the reciprocal basis as well as the elements of the dual basis serve as projection operators suggests that there exists a more far reaching relation between these two bases. That this is indeed the case is expressed by the following

I. Proposition: The image of e_k^+ under the metric is ω_k^k ; in other words, the metric establishes a one-to-one relationship between the elements of the reciprocal basis and those of the dual basis:

$$e_k^+ \mapsto \omega_k^k; \text{ what about } \sum_{i \neq k} \omega_i^k?$$

see p. 21.14

proof:

$$\left. \begin{aligned} g(e_k^+, \vec{x}) &= e_k^+ \cdot \vec{x} = e_k^+ \cdot \vec{e}_i \cdot x^i = x^k \\ \langle \omega_k^k | \vec{x} \rangle &= \langle \omega_k^k | e_i \rangle x^i = x^k \end{aligned} \right\} \forall \vec{x}$$

This holds for all vectors \vec{x} . Consequently,

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$$g(e_k^+, e_k^+) = "e_k^+ \cdot e_k^+" = \omega_k^k$$

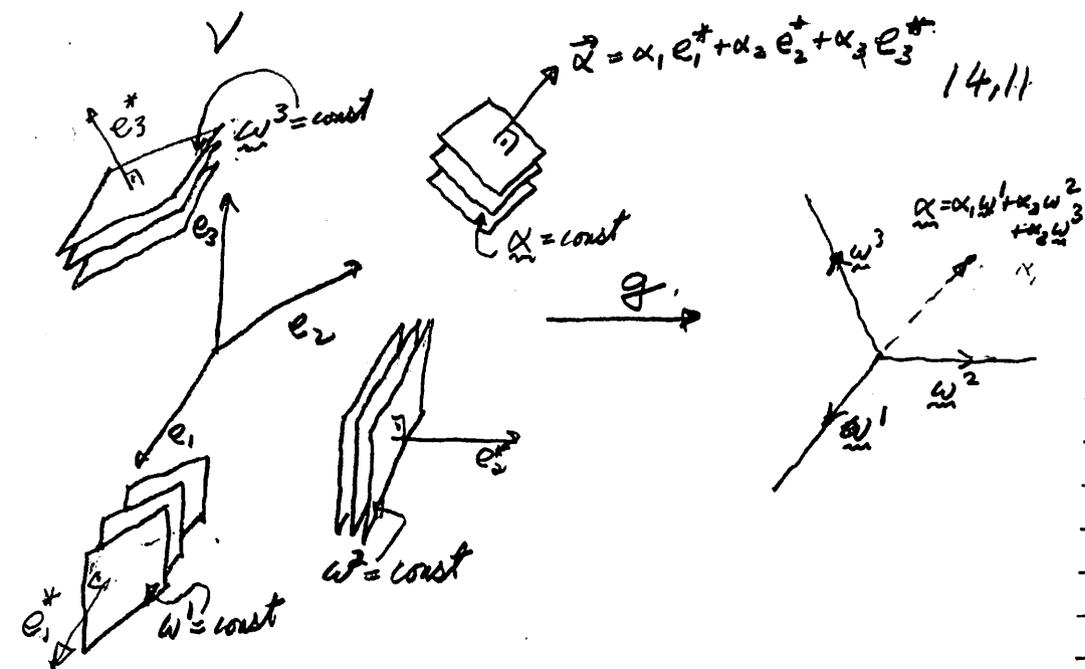
which is the same as

$$e_k^+ \mapsto g(e_k^+, e_k^+) = \omega_k^k \quad (**)$$

The thusly constructed linear function ω_k^k forces us into the conclusion that the vector e_k^+ is perpendicular to all the (parallel plane) level surfaces (= isograms) of ω_k^k . Indeed, recall that a vector is said to be perpendicular to a planar surface if it is perpendicular to every vector lying in this plane,

$$\langle \omega_k^k | e_i \rangle = 0, \quad (**)$$

which is the case for all $i \neq k$.



to the vector \vec{e}_k^* , i.e.

$$q(\vec{e}_k^*, \vec{e}_i) = \omega^k(\vec{e}_i) = (\omega^k | \vec{e}_i \rangle = 0 \quad (i \neq k)$$

Thus one is forced to conclude that

\vec{e}_k^* is a vector perpendicular to the isograms of ω^k . This vector is

unique because it satisfies the additional condition that

$$\vec{e}_k^* \cdot \vec{e}_k = 1 \quad (\text{no summation})$$

Having constructed the basis $\{\vec{e}_k^*\}$ reciprocal to $\{\vec{e}_i\}$, one now has the following Proposition

This perpendicularity is illustrated for each ω^k ($k=1, 2, 3$) on the left hand side of the above picture. In light of Eq. (*) at the top of P 14.10, the condition of \vec{e}_i ($i \neq k$) lying in the plane, namely Eq. (**), is equivalent to all these vectors \vec{e}_i being perpendicular

II Proposition

Given the linear function (covector)

$$\alpha = \alpha_R \omega^R$$

Conclusion:

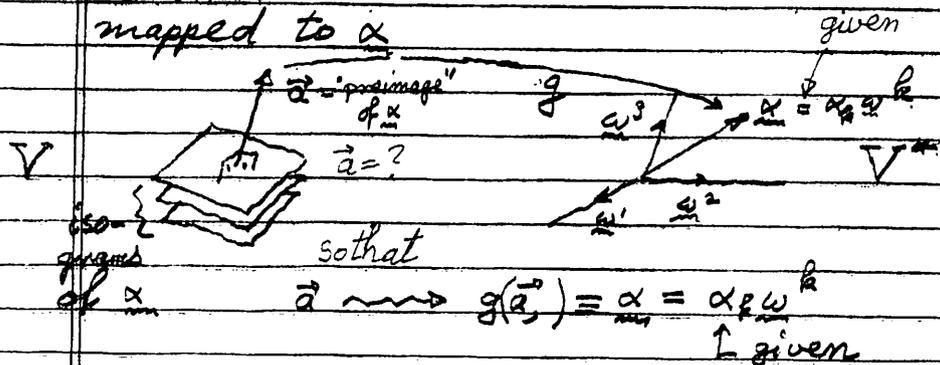
(a) The preimage under the metric g is the unique vector

$$\vec{a} = \sum_R \alpha_R e_R^*$$

(b) This vector is perpendicular to the isograms of α .

Discussion:

(a) "Preimage" of α is the set of vectors that get mapped to α . Here the preimage consists of a single element mapped to α .



(b) To validate that \vec{a} with the property that

$$g(\vec{a}, \cdot) = \alpha \quad (= \alpha_R \omega^R \text{ given})$$

is a vector which is \perp to the isograms of α , let \vec{x} be any vector in V . Then

$$\begin{aligned} \alpha(\vec{x}) &= \langle \alpha_R \omega^R | \vec{x} \rangle \\ &= \alpha_R \langle \omega^R | \vec{x} \rangle \\ &= \alpha_R x^R \\ &= \sum_R \alpha_R e_R^* \cdot \vec{x} \quad \forall \vec{x} \end{aligned}$$

This shows that

$$\vec{a} = \sum_R \alpha_R e_R^*$$

is the preimage of α .

Next let \vec{x} be any vector lying in the isogram where α has the value zero,

i.e. $\alpha(\vec{x}) = 0$

14.15

Consequently,

$$\vec{x} \in V_{\alpha} \equiv \{ \vec{x} : \alpha(\vec{x}) = 0 \} \quad \alpha = \sum_{\underline{m}} \alpha_{\underline{m}} \omega^{\underline{m}}$$

$$= \{ \vec{x} : \vec{a} \cdot \vec{x} = 0 \} \quad \vec{a} = \sum \alpha_{\underline{m}} \underline{e}_{\underline{m}}$$

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The fact that \vec{a} is perpendicular to all vectors \vec{x} lying in planar isogram $\alpha = 0$ is what is meant by the statement that \vec{a} is the vector perpendicular to all the level planar isograms of α . Why? Because they are parallel planes.

Conclusion:

$$\vec{a} = \sum \alpha_{\underline{m}} \underline{e}_{\underline{m}}$$

is the (unique) vector which is perpendicular to the isograms of

$$\alpha = \sum \alpha_{\underline{m}} \omega^{\underline{m}}$$

where $\left. \begin{array}{l} \langle \omega^{\underline{k}} | \underline{e}_i \rangle \\ \text{and} \\ \underline{e}_{\underline{k}} \cdot \underline{e}_i \end{array} \right\} = \delta_{\underline{k}i}$

An alternative validation of Proposition II on pages 14.13-14.15

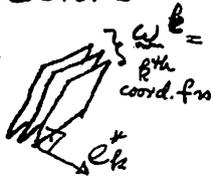
Recall that the identifying property of e_k^* is that it is orthogonal to $e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n$ and that $e_k^* \cdot e_k = 1$,

i.e.
$$e_k^* \cdot e_j = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases}$$

Thus e_k^* is perpendicular to the level surfaces of ω^k , i.e. perpendicular to those vectors \vec{x} which satisfy $0 = \omega^k(\vec{x}) = \langle \omega^k, \vec{x} \rangle$.

In other words

$$\langle \omega^k, \vec{x} \rangle = 0 \Rightarrow e_k^* \cdot \vec{x} = 0$$



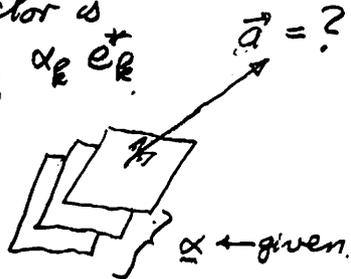
One can extend this orthogonality from ω^k to arbitrary linear functionals

$$\underline{\alpha} = \alpha_k \omega^k$$

Question: Is there a unique vector \perp to the level surfaces of $\underline{\alpha} = \alpha_k \omega^k$, and if so what is it?

Answer: Yes, this vector is

$$\vec{a} = \sum_{k=1}^n \alpha_k e_k^*$$



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Validation:

Let \vec{x} be any vector lying in the isogram ("level surface") $\underline{\alpha} = 0$ of the given $\underline{\alpha} = \alpha_k \omega^k$:

$$0 = \underline{\alpha}(\vec{x}) \equiv \langle \underline{\alpha}, \vec{x} \rangle$$

i.e. $\alpha_1 x^1 + \dots + \alpha_n x^n = 0$

(" \vec{x} does not pierce any isogram of $\underline{\alpha}$ ")

The vector $\vec{a} \in V$ corresponding to $\underline{\alpha} \in V^*$ is determined by the given metric

$$g: V \rightarrow V^*$$

$$\vec{a} \mapsto \underline{\alpha} = g(\vec{a}, \cdot) = \alpha_i \omega^i \quad (*)$$

Furthermore, the vector \vec{a} can be expanded in terms of the reciprocal basis vectors e_i^* :

$$\vec{a} = a^{+i} e_i^*$$

Q: What are the expansion coefficients?

A: They are determined by Eq(*):

$$\alpha_i \omega^i = g(a^{+j} e_j^*, \cdot)$$

$$= a^{+j} g(e_j^*, \cdot)$$

$$= a^{+j} \omega^j$$

defining property of e_i^* . See P21,13

$$\therefore a^{+i} = \alpha_i \text{ for } i = 1, \dots, n.$$

Conclusion:

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~~12.11~~

$$\vec{a} = \alpha_i e_i^* \rightsquigarrow \alpha = \alpha_i \omega_i$$

Furthermore, any vector \vec{x} in the isogram $\alpha = 0$,

satisfies

$$0 = \alpha(\vec{x}) = g(\vec{a}, \vec{x}) = \vec{a} \cdot \vec{x}$$

In other words, $\vec{a} \perp$ to the isogram $\alpha = 0$

Q.E.D.

EXTRA APPLICATION
 Bragg Diffraction expressed in terms of Laue's Eq'n

Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be basis ("primitive translation" vectors) for a crystal lattice V

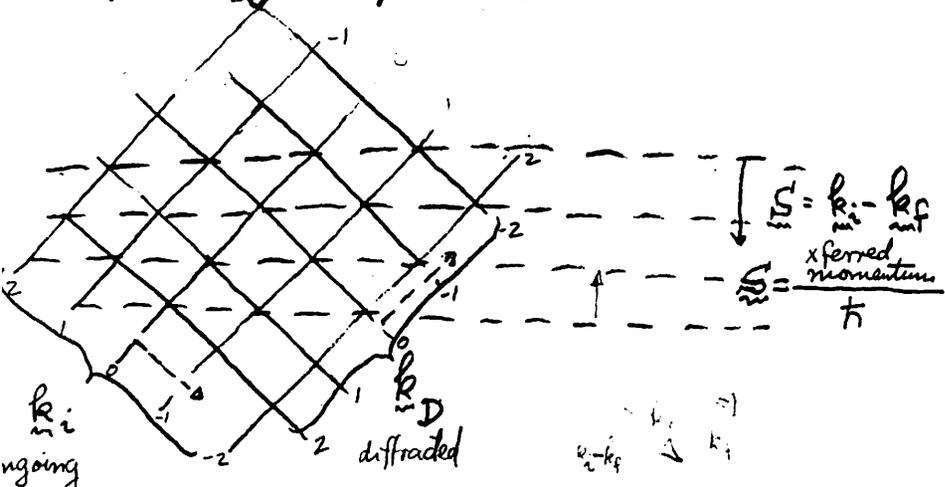
Let $\{\omega^1, \omega^2, \omega^3\}$ be the basis of duals: $\langle \omega^i, e_j \rangle = \delta^i_j$

Let $\underline{k}_i =$ incident propagation covector for the wave $\exp(i \langle \underline{k}_i, \vec{x} \rangle) = \exp[i(k_1 x^1 + k_2 x^2 + k_3 x^3)]$

Let $\underline{k}_D =$ diffracted propagation covector

Let $\underline{S} = \underline{k}_i - \underline{k}_D$ [note: $(\underline{k}_i - \underline{k}_D)\hbar =$ momentum transferred to the crystal]

Let h, k, l be integers, the "Miller indices" of a crystal plane



Then an incident wave $e^{i \langle \underline{k}_i, \vec{x} \rangle}$ will get diffracted by the crystal lattice into a diffracted wave $e^{i \langle \underline{k}_D, \vec{x} \rangle}$ (i.e. Bragg diffraction occurs) provided

$$\begin{aligned} \langle \underline{S}, \vec{e}_1 \rangle &= 2\pi h = \text{phase along } \vec{e}_1 \\ \langle \underline{S}, \vec{e}_2 \rangle &= 2\pi k = \text{phase along } \vec{e}_2 \\ \langle \underline{S}, \vec{e}_3 \rangle &= 2\pi l = \text{phase along } \vec{e}_3 \end{aligned}$$

These are Laue's eq'ns. They can be restated

$$\frac{1}{2\pi} \underline{S} = h \omega^1 + k \omega^2 + l \omega^3$$

$h, k, l = \text{integers}$

Laue's eq'n for Bragg diffraction

This eq'n says that Bragg diffraction \iff level surfaces of \underline{S} must coincide with some crystal plane (h, k, l)

For details see Charles Kittel: "Introduction to Solid State Physics."

$$\frac{1}{2\pi} \vec{S} = h \vec{e}_1^* + k \vec{e}_2^* + l \vec{e}_3^* \in \text{Reciprocal Lattice}$$

Lattice of crystal planes, used in X-Ray crystallography & quantum theory of solids.

Tensors as Multilinear Maps.

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By generalizing linear and bilinear maps one arrives at the idea of a tensor. In fact, a tensor is a multilinear map. The idea of multilinearity is illustrated by the following

Example (The determinant)

Consider the determinant each of whose rows is the set of components of a vector

$$\vec{A}_1 : A_1^1, A_1^2, \dots, A_1^n$$

$$\vec{A}_2 : A_2^1, A_2^2, \dots, A_2^n$$

$$\vec{A}_n : A_n^1, A_n^2, \dots, A_n^n$$

This determinant,

$$\det(\vec{A}_1, \dots, \vec{A}_n) = \begin{vmatrix} A_1^1 & \dots & A_1^n \\ \vdots & & \vdots \\ A_n^1 & \dots & A_n^n \end{vmatrix}$$

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is a multilinear map because

$$\det(\dots, \alpha \vec{A}_i + \beta \vec{B}_i, \dots) = \alpha \det(\dots, \vec{A}_i, \dots) + \beta \det(\dots, \vec{B}_i, \dots)$$

The most general definition of a multilinear map does not make any stipulations about the dimensionality of each vector, nor about the choice of basis for each vector space,

15.3

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Tensors: Their Definition

Read Ch 3 of Grant Read MTW 3.2, Box 3.2, 3.5 also 4.2-4.3

TENSORS and MULTILINEAR ALGEBRA

By generalizing bilinear maps to multilinear maps one arrives at the idea of a tensor. In fact a multilinear map is a tensor. This definition is completely independent of the choice of any basis.

Definition: (multilinearity)

Let V_1, V_2, \dots, V_q be vector spaces. The map

$$H: V_1 \times V_2 \times \dots \times V_q \rightarrow R$$

$$(v_1, v_2, \dots, v_q) \mapsto H(v_1, v_2, \dots, v_q)$$

is said to be multilinear if

$$H(v_1, \dots, \alpha v_i + \beta w_i, \dots, v_q) = \alpha H(v_1, \dots, v_i, \dots, v_q) + \beta H(v_1, \dots, w_i, \dots, v_q) \quad \forall 1 \leq i \leq q.$$

Definition: (tensor)

Let $V_1 = V_2 = \dots = V_m = V^*$
 $V_{m+1} = \dots = V_{m+n} = V$

then the multilinear map

$$H: \underbrace{V^* \times V^* \times \dots \times V^*}_n \times \underbrace{V \times V \times \dots \times V}_m \rightarrow R$$

$$\underbrace{(\sigma_1, \sigma_2, \dots, \sigma_n)}_{n \text{ covectors}} \times \underbrace{(u_1, u_2, \dots, u_m)}_{m \text{ vectors}} \mapsto H(\sigma_1, \sigma_2, \dots, \sigma_n, u_1, u_2, \dots, u_m)$$

is a tensor of rank $\binom{n}{m}$ \leftarrow "contravariant" rank \leftarrow "covariant" rank

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Tensors: Examples

Note about notation: We shall also use bold face symbols (instead of only arrowed ones) to denote vectors. Thus $\vec{u} = \mathbf{u}$.

Examples of tensors

	Symbol	Mapping	Rank
covector ①	ω	$V \rightarrow R$ $V \mapsto \omega(V) = \langle \omega V \rangle$	$\binom{0}{1}$
metric ②	g	$V \times V \rightarrow R$ $(u, v) \mapsto g(u, v) = u \cdot v$	$\binom{0}{2}$
vector ③	w	$V^* \rightarrow R$ $\sigma \mapsto w(\sigma) = \langle \sigma w \rangle$	$\binom{1}{0}$
inverse metric ④	g^{-1}	$V^* \times V^* \rightarrow R$ $(f, h) \mapsto g^{-1}(f, h) = \langle f \vec{h} \rangle = \vec{h}_f \cdot \vec{h}_h$	$\binom{2}{0}$