

Lecture 16

Coordinate components of a tensor

Tensor product

Tensor product basis; examples

Tensor space

New tensors via

- (i) lowering indices
- (ii) contractions

Examples of tensors

More examples:

(i) Levi Civita tensor

(ii) Exterior product

(iii) Particle flux tensor

[MTW 3,5; 4,1-4,2]

for Lecture 17

16.1

Coordinate Components of a Tensor

A vector has components relative to a given basis. So does a covector. This concept, "components relative to a given basis," can be extended to tensors. The basis

dependent coordinate components of a tensor are obtained by projecting them out with the basis elements as follows:

Definition (Tensor components relative to a given basis)

Let $\{e_i\}$ be a basis for V
Let $\{w^i\}$ be its dual basis for V^*

Then the numbers

$$H(w^{j_1}, w^{j_2}, \dots, w^{j_m}, e_{i_1}, e_{i_2}, \dots, e_{i_n}) = H^{j_1 j_2 \dots j_m i_1 i_2 \dots i_n}$$

are the components of B_l relative to

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the given basis.

Example:

$$\alpha(e_k) = \alpha^i w^k(e_i) = \underbrace{\alpha^i}_{S^3_k} \underbrace{w^k}_{S^3_k}$$

$$\begin{aligned} u(w^k) &= u^i e_i(w^k) \\ &= u^i w^k(e_i) = \underbrace{u^i}_{S^3_k} \underbrace{w^k}_{S^3_k} = u^k \end{aligned}$$

Comment:

The result of evaluating H on some arbitrary $n+m$ tuple $(\beta_1, \dots, \beta_m, u_1, \dots, u_n, w^1, \dots, w^m)$ is

$$\begin{aligned} H(\beta_1 w^{j_1}, \dots, \beta_m w^{j_m}, u^i e_i, \dots, u^n e_n, w^1 e_1, \dots, w^m e_m) &= \\ &= H^{j_1 j_2 \dots j_m i_1 i_2 \dots i_n} (\beta_1 \dots \beta_m u^i \dots u^n w^m) \end{aligned}$$

which is an $n+m$ fold sum.

It is worth while to remind ourselves that

- (a) the distinction of upper vs lower indices is to be observed with rigid rigour;
- (b) the summation ("dummy") indices appear only in pairs.

How to construct a tensor.

Tensor product

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By taking an appropriately defined product
(the "tensor product") of linear maps
one can obtain a multilinear map.
This is done as follows:

Definition (Tensor Product)

Let $a, b, \dots, c \in V$
 $\alpha, \beta, \dots, \gamma \in V^*$

The multilinear map (functional)
 $a \otimes b \otimes \dots \otimes c \otimes \alpha \otimes \beta \otimes \dots \otimes \gamma : V \times V \times \dots \times V \times V \times \dots \times V \rightarrow R$

$(\underbrace{a_1, a_2, \dots, a_n}_{n}, \underbrace{b_1, b_2, \dots, b_m}_{m}, \dots, \underbrace{\gamma_1, \gamma_2, \dots, \gamma_n}_{n}) \mapsto \langle a_1 | \alpha \rangle \langle a_2 | \beta \rangle \dots \langle a_n | \gamma \rangle$

is the tensor product of a, \dots, γ .

Their

Basic Representation:

Having generated (n) rank tensors by taking tensor products, one can take linear combinations of such products to obtain any tensor. In fact a linear combination of tensor products of basis elements yields the basis representation of a (n) rank tensor.

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This idea is captured in the following
Proposition: (Basis Representation of a Tensor)

Given (i) a basis $\{e_i\}$ of V
(ii) the tensor H of rank (n)

Conclusion:

$$H = H^{j_1 \dots j_n} e_{j_1} \otimes \dots \otimes e_{j_n} \otimes \omega^{i_1} \otimes \dots \otimes \omega^{i_m}$$

This is the representation of H in term of the basis and its dual.

Proof: One must show that the value of the linear map on the l.h.s. equals the value of the linear map on the r.h.s. for all $n+m$ tuples of covectors and vectors.

It is sufficient to show this for the basis of $n+m$ tuples
 $(\omega^{j_1}, \dots, \omega^{j_n}, \epsilon_{i_1}, \dots, \epsilon_{i_m})$

and then recall the definition of Tensor components relative to a given basis:

$$H(\omega^{j_1}, \dots, \omega^{j_n}, \epsilon_{i_1}, \dots, \epsilon_{i_m}) = H^{j_1 \dots j_n i_1 \dots i_m} e_{j_1} \otimes \dots \otimes e_{j_n} \otimes \omega^{i_1} \otimes \dots \otimes \omega^{i_m} (- - -)$$

Examples

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- Metric tensor:

$$\textcircled{1} \quad g = g_{ij} \underline{\omega^i} \otimes \underline{\omega^j} = g_{ij} dx^i \otimes dx^j$$

Inverse metric tensor: Later For $\underline{\omega^i} = \underline{dx^i}$

$$\text{inverse metric} \quad \textcircled{2} \quad g^{-1} = g^{ij} e_i \otimes e_j = g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

$$\text{Comment: } g^{ij}(dS, dS) \equiv g^{ij} \langle dS | e_i \rangle \langle dS | e_j \rangle$$

$$= g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = \nabla S \cdot \nabla S$$

could be called the "H-J operator"
 (3) Cartan's unit tensor " dP "

$$"dP" = \underline{\omega^i} \otimes e_i = \delta^i_j \underline{\omega^j} \otimes e_i$$

GO TO 4b 16.6 (Levi-Civita tensor)

(4) The totally antisymmetric (Levi-Civita) tensor in n dimensions expanded relative to a chosen basis $\{\underline{\omega^i}\}$ is

$$E = \epsilon_{i_1 \dots i_n} \underline{\omega^{i_1}} \otimes \dots \otimes \underline{\omega^{i_n}}$$

OBSOLETE

Here $\epsilon_{i_1 \dots i_n}$ is totally antisymmetric,
 in other words,

(5) Particle flux tensor (or P 16.10)

SKIP Goto 16.6b

incomplete:
 $E_{12\dots n} = \sqrt{g}$ is missing.

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$E_{i_1\dots i_m} = 0$ if any pair of indices are the same

$= E_{i_1\dots i_m}$ if $i_1\dots i_m$ is an even permutation of
 $i_1\dots i_n$

$= -E_{i_1\dots i_m}$ if $i_1\dots i_m$ is an odd permutation of
 $i_1\dots i_n$

This is a frame dependent definition

An equivalent but frame ("basis") independent definition is:

$$E(\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n) = \det \begin{vmatrix} \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \dots & \omega^n(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \dots & \omega^n(\vec{A}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \omega^1(\vec{A}_n) & \omega^2(\vec{A}_n) & \dots & \omega^n(\vec{A}_n) \end{vmatrix} = \text{volume of parallelopiped subtended by } \{\text{the image of } \vec{A}_i\} \text{ in } \mathbb{R}^n$$

Example 4b (Levi Civita)

The "Volume tensor," or "Levi Civita" tensor is a totally antisymmetric tensor.

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$E : V \times V \times \dots \times V \rightarrow \mathbb{R}$ with Rank = $\binom{0}{n}$

$$(\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n) \text{ and } E(\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n) = \det |$$

where $E(\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n)$ changes

sign when any two of the vectors are interchanged, and

$$E(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = \begin{cases} + |\det g|^{1/2} & \text{if } (e_1, \dots, e_n) \text{ are positively oriented} \\ - |\det g|^{1/2} & \text{if } (e_1, \dots, e_n) \text{ are negatively oriented.} \end{cases}$$

(Jacobian from O.N. to oblique coordinates)

Here $\det g$ is the determinant of the metric g and the + or - sign expresses the orientation (positive or negative) of the basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$. One can show that

$E(\vec{e}_1, \dots, \vec{e}_n) = (\text{oriented}) \underline{\text{volume}}$
spanned by the parallelopiped $(\vec{e}_1, \dots, \vec{e}_n)$