Lecture 17

1. The volume (Levi-Civita) tensor
2. Exterior product [MTW 3.5; 4.1-4.2]
3. Particle Flux (2) Tensor
   in Euclidean space: Flux tube structure
   [Read Box 4.2, Fig 4.2-4.5, Box 4.4]

Supplement

The Particle (or Charge) Density-Flux 3-form
a.k.a. the Worldline Density (3) Tensor
Example 4b (Levi-Civita)

The "Volume tensor" or Levi-Civita's totally antisymmetric tensor:

\[ \mathbf{E} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R} \quad \text{with} \quad \text{Rank} = (0) \]

\[ (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \rightarrow \mathbf{E}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = \det | \begin{array}{cccc} e_1^1 & e_1^2 & \cdots & e_1^n \\ e_2^1 & e_2^2 & \cdots & e_2^n \\ \vdots & \vdots & \ddots & \vdots \\ e_n^1 & e_n^2 & \cdots & e_n^n \end{array} | \]

where \( \mathbf{E}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \) changes sign when any two of the vectors are interchanged, and

\[ \mathbf{E}((\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)) = \begin{cases} + |\det g|^{1/2} & \text{if } (i_1, \ldots, i_n) \text{ is positively oriented} \\ - |\det g|^{1/2} & \text{if } (i_1, \ldots, i_n) \text{ is negatively oriented} \end{cases} \]

where \( |\det g| \) is the determinant of the oriented metric \( g \) and the \( + \) or \( - \) sign expresses the orientation (positive or negative) of the basis.

One can show that

\[ \mathbf{E}(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \text{(oriented) volume spanned by the parallelepiped } (\mathbf{e}_1, \ldots, \mathbf{e}_n) \]

in order to impart to \( \mathbf{E} \) the fact that \( \mathbf{E}(\mathbf{e}_1, \ldots, \mathbf{e}_n) \) is the oriented volume spanned by \( \mathbf{e}_1, \ldots, \mathbf{e}_n \).

The basis expansion of \( \mathbf{E} \) is given by

\[ \mathbf{E} = E_{i_1 \cdots i_n} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_n} \]

Here \( E_{i_1 \cdots i_n} \) is the totally antisymmetric Levi-Civita symbol

\[ E_{i_1 \cdots i_n} = 0 \quad \text{if any pair of indices are the same} \]

\[ = E_{i_1 \cdots i_n} \quad \text{if } (i_1, \ldots, i_n) \text{ is an even permutation of } 1, \ldots, n \]

\[ = -E_{i_1 \cdots i_n} \quad \text{if } (i_1, \ldots, i_n) \text{ is an odd permutation of } 1, \ldots, n \]

and

\[ E_{i_2 \cdots i_n}^{i_1} = \begin{cases} + |\det g|^{1/2} & \text{if } (i_1, i_2, \ldots, i_n) \text{ is positively oriented} \\ -|\det g|^{1/2} & \text{if } (i_1, i_2, \ldots, i_n) \text{ is negatively oriented} \end{cases} \]
The basis expansion of $\varepsilon$ is given by

$$\varepsilon = \varepsilon_{i_1\ldots i_m} \omega^{i_1} \otimes \cdots \otimes \omega^{i_m}$$

Here $\varepsilon_{i_1\ldots i_m}$ is the totally antisymmetric Levi-Civita symbol.

$$\varepsilon_{i_1\ldots i_m} = 0 \text{ if any pair of indices are the same}$$

$$= \varepsilon_{i_1\ldots i_m} \text{ if } (i_1,\ldots, i_m) \text{ is an even permutation of } 1,\ldots, n$$

$$= -\varepsilon_{i_1\ldots i_m} \text{ if } (i_1,\ldots, i_m) \text{ is an odd permutation of } 1,\ldots, n$$

and

$$\varepsilon_{i_1\ldots i_m} = \begin{cases} \sqrt{\text{det} g} & \text{positive orientation of } g_{i_1\ldots i_m} \\ -\sqrt{\text{det} g} & \text{negative orientation of } g_{i_1\ldots i_m} \end{cases}$$

in order to impart to $\varepsilon$ the fact that $\varepsilon(\vec{v}_1,\ldots, \vec{v}_m)$ is the oriented volume spanned by $\vec{v}_1,\ldots, \vec{v}_m$. 
Using the Levi-Civita symbols, we notice that this volume can also be written as

\[\epsilon(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \epsilon_{123} \omega^1(a) \omega^2(b) \omega^3(c)\]

Specializing to \(n=3\) for illustrative purposes, one obtains:

\[\epsilon(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \epsilon_{12345} \omega^1(a) \omega^2(b) \omega^3(c) \omega^4(d)\]

In \(n\) dimensions \(\epsilon\) is

\[\epsilon = \frac{1}{n!} \epsilon_{12...n} \omega^1 \omega^2 ... \omega^n\]

In \(n=2\) dimensions one has

\[\epsilon = \epsilon_{ij} \omega^i \omega^j\]

This is a totally antisymmetric tensor of rank 3. With this definition in place, the Levi-Civita tensor for the \(n=3\) dimensional space is

\[\epsilon_{123} \omega^1 \omega^2 \omega^3\]

\[= \frac{1}{2} \left( \epsilon_{12} \omega^1 \omega^2 + \epsilon_{13} \omega^1 \omega^3 + \epsilon_{23} \omega^2 \omega^3 \right)\]
Example 6 (Particle Flux Tensor)

The *Particle Flux tensor* arises as follows:

The velocity of a fluid in three dimensional Euclidean space gives rise to its flux, an antisymmetric tensor of rank 2.

Let \( \mathbf{v} \) be the uniform velocity of the fluid, and let \( N \) be the density of the particles that make up the fluid.

Consider the area of the parallelogram spanned by the pair of vectors \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \), the number of particles per unit time crossing this area is

\[
|\det g| \text{[} N \mathbf{v} (\omega_1 \times \omega_2 - \omega_2 \times \omega_1) + N \mathbf{v} (\omega_1 \times \omega_2 - \omega_2 \times \omega_1) \text{]} (\mathbf{A}_1 \times \mathbf{A}_2) = \mathbf{g} (\mathbf{A}_1, \mathbf{A}_2)
\]

Introducing the exterior product

\[
\omega_1 \wedge \omega_2 = \omega_1 \times \omega_2 - \omega_2 \times \omega_1
\]

and the Levi-Civita tensor component

\[
\epsilon_{123} = |\det g|^{1/2} \quad \text{"positive orientation"}
\]

one obtains the particle flux tensor,

\[
\mathbf{g} = N \left( \epsilon_{123} \mathbf{v} \omega_1 \omega_2 + \epsilon_{231} \mathbf{v} \omega_2 \omega_1 + \epsilon_{312} \mathbf{v} \omega_3 \omega_2 \right)
\]

or

\[
\mathbf{g} = N \frac{1}{2!} \mathbf{v} \epsilon_{ijk} \omega_i \omega_j \omega_k = N \epsilon (\mathbf{\omega})
\]

\( \text{\# of particles flowing per unit time per (cross) oriented area.} \)
Let us choose a basis \{e_1, e_2, e_3\} relative to which \( V = v_1 e_1 + v_2 e_2 + v_3 e_3 \), i.e., the first basis vector is lined up with \( V \).

In that case

\[
\mathcal{F} = N \, v^1 \, e_{13} \, \omega^2 \, \omega^3
\]

This is a scalar valued two-form. This expression leads to the picture of the bilinear map \( \mathcal{F} \) as a flux tube structure whose value

\[
\mathcal{F} (\vec{A}_1, \vec{A}_2) = N \, v^1 \, e_{13} \, (A_1^2 A_3^3 - A_2^3 A_3^2)
\]

equals \( N \, s (\vec{A}_1, \vec{A}_2) \) the number of flux tubes intercepted by the parallelogram \( (\vec{A}_1, \vec{A}_2) \).

The essential feature of the concept of the \( (2) \)-rank tensorial flux-tube structure, which in the previous pages was developed for a 3-dimensional vector space, remains the same for a 4-dimensional vector space. This is done in Box 4.2 and Figures 4.2-4.5 of MTW.