

# Lecture 18

1. Example of a tensor of rank(0<sub>3</sub>):

The Particle (or charge) Flux-Density  
"3-form".

2. Tensor Space: New Tensors via addition

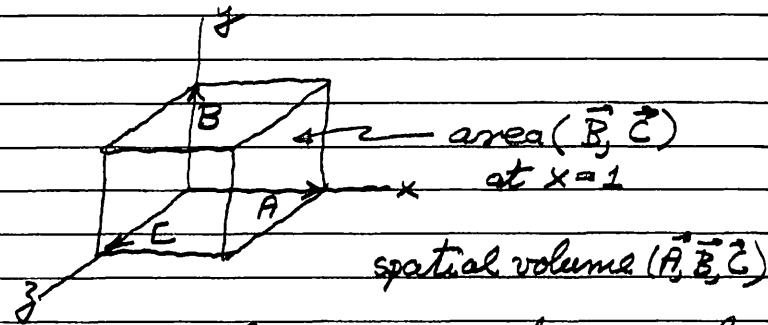
3. New Tensors via "Raising & Lowering  
Indices".

4. New Tensors via contraction.

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Example 7 (Particle density flux;  $\alpha^{(3)}$  tensor on spacetime)

Consider the flow of particle across from a box whose volume is spanned by vectors  $\vec{A}, \vec{B}$ , and  $\vec{C}$ .



We consider the circumstance where all particles have the same velocity, along the  $x$ -direction and moving in the fixed plane, say,  $z=7$ .

At  $t=0$ , say three of them start at  $x=0$ , while another three start at say  $x=.5$ .

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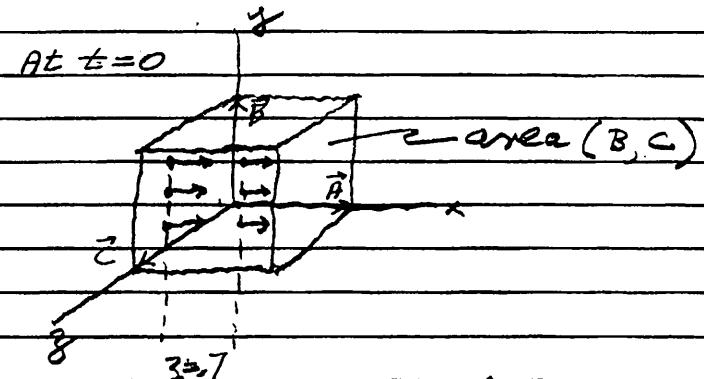


Figure 1: Snapshot at  $t=0$  of particles in a box.

These six particle will ultimately pass through the area  $(B, C)$ , the three particles from the middle ( $x=.5$ ) first, the particles from  $x=0$  last.

Question: What is the space time picture of the world lines of these six particles?

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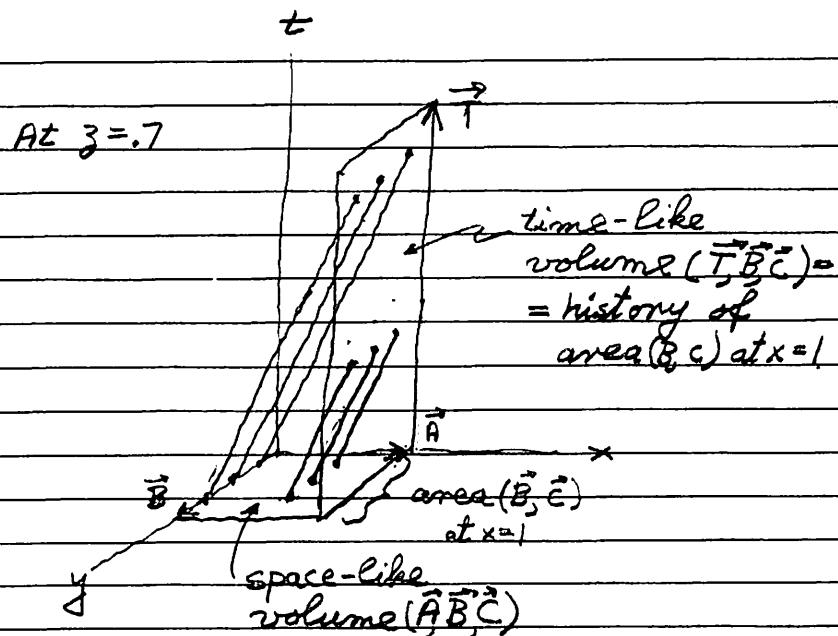


Figure 2: Worldline (= histories) of particles having the common four-velocity  $\nu$  in the time-like 3-D hypersurface  $\beta = .7$

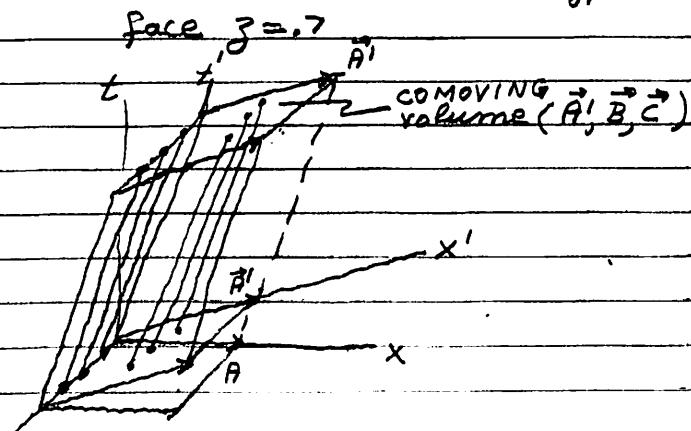


Figure 3: In the comoving volume  $(\vec{A}',\vec{B},\vec{C})$  the particles have zero spatial velocity components

$$\frac{dx'}{dt} = 0$$

$$\frac{dy'}{dt} = 0$$

$$\frac{d\vec{z}'}{dt} = 0$$

In the comoving frame  $(t',x',y'=y,\vec{z}'=\vec{z})$  the particles have zero spatial velocity and their comoving ("proper") density is

$$N = \frac{\#}{\text{proper volume } (\vec{A}',\vec{B},\vec{C})} \quad \begin{matrix} \text{"proper"} \\ \text{particle} \\ \text{density"} \end{matrix}$$

Let the four-velocity  $u$  of these particles be the same. In particular, have its lab frame components be

$$(1) \quad \{u^0, u^1, u^2, u^3\} = \left\{ \frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0 \right\} = \left\{ 8, 8 \frac{dx}{d\tau}, 0, 0 \right\}$$

Focus on a pipe segment whose volume is spanned by the <sup>(spatial)</sup> vectors  $\vec{A}, \vec{B}$ , and  $\vec{C}$ , and consider the motion of particles, say  $\#$  of them, which at  $t=0$  start their motion in

volume  $(\vec{A}, \vec{B}, \vec{C})$ . The <sup>18-6</sup> LAB density of these particles in this volume is

$$\boxed{N_{LAB} = \frac{\#}{vol(\vec{A}, \vec{B}, \vec{C})|_{t=0}}} \quad (2)$$

With time as a fourth coordinate, these particles have parallel worldlines with common tangent, and at  $t=0$  they pass through volume  $(\vec{A}, \vec{B}, \vec{C})$ .

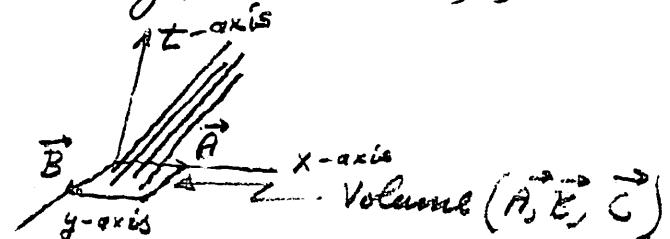


Figure 18.4 : Worldlines of  $\#=4$  particles at  $t=0$  passing through volume  $(\vec{A}, \vec{B}, \vec{C})$ . Note that reference to vector  $\vec{C}$  has been suppressed.

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2) The fact that the flowing particles move along the pipe with lab  $x$ -velocity  $\{\frac{dx}{dt}, 0, 0, 0\}$  implies that a certain number of them, namely  $N_{LAB} \frac{dx}{dt} \Delta t \cdot \text{area}(\vec{B}, \vec{C})$

of them are crossing the area  $(\vec{B}, \vec{C})$  located at  $x=1$  (see Fig. 14.2) during lab time interval  $\Delta t = T$

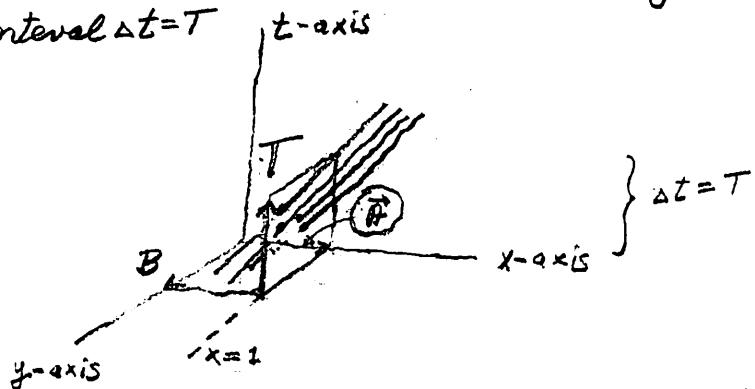


Figure 14.2 Worldlines of # particles at  $x=1$  crossing area  $(\vec{B}, \vec{C})$  during time  $\Delta t = T$ , or equivalently, passing through the time-like volume  $(T, \vec{B}, \vec{C})$  located at  $x=1$ . Here  $T$  is the time-like vector whose LAB components are  $\{\Delta t = T, 0, 0, 0\}$ ,

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This time interval together with the cross sectional area form a 3-dimensional volume

$$\text{vol}(T, \vec{B}, \vec{C}) = \Delta t \cdot \text{area}(\vec{B}, \vec{C})$$

and the worldlines of the above-mentioned  $N_{LAB} \frac{dx}{dt} \Delta t$  particles pass through this volume at  $x=1$ . Consequently,

$$N_{LAB} \frac{dx}{dt} \underbrace{\Delta t \cdot \text{area}(\vec{B}, \vec{C})}_{\text{vol}(T, \vec{B}, \vec{C})} = \#$$

or

$$\boxed{N_{LAB} \frac{dx}{dt} = \frac{\#}{\text{vol}(T, \vec{B}, \vec{C})|_{x=1}} = \frac{(\text{particles})}{(\text{time})(\text{area})}}, \quad (3)$$

which is the particle flux into the  $x$ -direction in the LAB frame.

3.) We now combine Eqs(1), (2), and (3) into a single geometrical concept.

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18-5

We do this by introducing the quantity  $N$ , the proper (=comoving) particle density by means of the equation

$$N_{LRB} \equiv N \gamma = Nu^0.$$

Using Eq.(4), one finds that as a consequence Eqs(2) and (3) become

$$Nu^0 = \frac{\#}{\text{vol}(\vec{A}, \vec{B}, \vec{C})} \Big|_{t=0} \quad (4)$$

$$Nu' = \frac{\#}{\text{vol}(\vec{I}, \vec{B}, \vec{C})} \Big|_{x=1} \quad (5)$$

We see that the quantity  $N$  is the comoving (proper!) particle density indeed!

- 4.) By rewriting Eqs (4)&(5) in the form

$$Nu^0 \cdot \text{vol}(\vec{A}, \vec{B}, \vec{C}) \Big|_{t=0} = \#$$

$$Nu' \cdot \text{vol}(\vec{I}, \vec{B}, \vec{C}) \Big|_{x=1} = \#$$

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and replacing  $(\vec{A}, \vec{B}, \vec{C})$  and  $(\vec{I}, \vec{B}, \vec{C})$  with a triple of generic four-vectors  $(\vec{A}, \vec{B}, \vec{C})$ , we introduce the following "scalar valued 3-form," an antisymmetric tensor of rank  $\binom{0}{3}$ , by means of the following equation

$$|\det g_{\mu\nu}|^{1/2} \begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \langle \omega^0 | A \rangle & \langle \omega^1 | A \rangle & \langle \omega^2 | A \rangle & \langle \omega^3 | A \rangle \\ \langle \omega^0 | B \rangle & \langle \omega^1 | B \rangle & \langle \omega^2 | B \rangle & \langle \omega^3 | B \rangle \\ \langle \omega^0 | C \rangle & \langle \omega^1 | C \rangle & \langle \omega^2 | C \rangle & \langle \omega^3 | C \rangle \end{vmatrix} \stackrel{*}{=} J(A, B, C) = \#$$

The value of this trilinear function  ${}^*J$  equals the number of particles found in  $\text{vol}(\vec{A}, \vec{B}, \vec{C})$ .

#7  
18-11

Using the exterior product of Example 5

on pages 16.8-16.9 one can write the

determinantal value  $\#$  on the previous

page as follows

$$\# = \star J(A, B, C) =$$

$$N u^0 \begin{vmatrix} \langle \omega^1 | A \rangle & \langle \omega^2 | A \rangle & \langle \omega^3 | A \rangle \\ \langle \omega^1 | B \rangle & \langle \omega^2 | B \rangle & \langle \omega^3 | B \rangle \\ \langle \omega^1 | C \rangle & \langle \omega^2 | C \rangle & \langle \omega^3 | B \rangle \end{vmatrix} \times |\det g_{\mu\nu}|^{1/2}$$

$$-N u^1 \begin{vmatrix} \langle \omega^0 | A \rangle & \langle \omega^2 | A \rangle & \langle \omega^3 | A \rangle \\ \langle \omega^0 | B \rangle & \langle \omega^2 | B \rangle & \langle \omega^3 | B \rangle \\ \langle \omega^0 | C \rangle & \langle \omega^2 | C \rangle & \langle \omega^3 | C \rangle \end{vmatrix} \times |\det g_{\mu\nu}|^{1/2}$$

$$+ N u^2 \begin{vmatrix} \langle \omega^0 | A \rangle & \langle \omega^1 | A \rangle & \langle \omega^3 | A \rangle \\ \langle \omega^0 | B \rangle & \langle \omega^1 | B \rangle & \langle \omega^3 | B \rangle \\ \langle \omega^0 | C \rangle & \langle \omega^1 | C \rangle & \langle \omega^3 | C \rangle \end{vmatrix} \times |\det g_{\mu\nu}|^{1/2}$$

$$-N u^3 \begin{vmatrix} \langle \omega^0 | A \rangle & \langle \omega^1 | A \rangle & \langle \omega^2 | A \rangle \\ \langle \omega^0 | B \rangle & \langle \omega^1 | B \rangle & \langle \omega^2 | B \rangle \\ \langle \omega^0 | C \rangle & \langle \omega^1 | C \rangle & \langle \omega^2 | C \rangle \end{vmatrix} \times |\det g_{\mu\nu}|^{1/2}$$

Each of these  $3 \times 3$  determinants has theform  $\omega^i \wedge \omega^j \wedge \omega^k (A, B, C) =$ 

$$= \{\omega^i \otimes \omega^j \otimes \omega^k + \omega^j \otimes \omega^k \otimes \omega^i + \omega^k \otimes \omega^i \otimes \omega^j\} (\star)$$

$$- \omega^k \otimes \omega^j \otimes \omega^i - \omega^i \otimes \omega^k \otimes \omega^j - \omega^j \otimes \omega^i \otimes \omega^k\} (A, B, C)$$

Consequently, using the defined Levi-Civita components on p 16.7, one finds

$$\# = \star J(A, B, C) =$$

$$= \{N u^0 E_{0123} \omega^1 \wedge \omega^2 \wedge \omega^3 + N u^1 E_{1023} \omega^0 \wedge \omega^2 \wedge \omega^3 + N u^2 E_{2013} \omega^0 \wedge \omega^1 \wedge \omega^3 + N u^3 E_{3012} \omega^0 \wedge \omega^1 \wedge \omega^2\} (A, B, C)$$

The antisymmetry of  $E_{\alpha\beta\gamma\delta}$  under the interchange of any pair of its indices combinedwith the antisymmetry of  $\omega^\alpha \wedge \omega^\beta$  under the interchange of any pair of its indices

(see P 16.8), results in

$$\# = \star J(A, B, C) = \frac{1}{3!} N u^0 E_{\alpha\beta\gamma\delta} \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta (A, B, C)$$

$$= \sum_{\alpha} \sum_{\beta<\gamma<\delta} N u^0 E_{\alpha\beta\gamma\delta} \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta (A, B, C)$$

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By referring to Eq. (#) on the previous page one also write this result in the form

$$\# = J(ABC) = N u^\alpha \epsilon_{\alpha\beta\gamma} w^\beta w^\gamma w^\delta (ABC)$$

In summary one concludes that

$$* J = \frac{1}{3!} N u^\alpha \epsilon_{\alpha\beta\gamma} w^\beta w^\gamma w^\delta$$

$$J = N u^\alpha \epsilon_{\alpha\beta\gamma} w^\beta w^\gamma w^\delta$$

is a tensor of rank (0). In physics it called the particle (or charge) density-flux

3-form. The scalar  $N$  is the proper (i.e. comoving) particle (or charge) density.

Mathematically it is a scalar-valued 3-linear map whose values yield the number of particles (or charges).

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This rank (0) tensor is the charge density flux of the Maxwell field equations recast in geometrical form as exhibited in § 4.3 and in Box 4.4 of MTW on p119.

It is applied to a highly stylized formulation (on p370) in Box 15.1 H), whose purpose is to highlight the parallelism between the Maxwell field equations and the Einstein field equations.

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# # #

A conspicuous feature of  ${}^*J$ , Eq.(\*) on p14.9

$${}^*J = \frac{1}{3} Nu^\alpha \epsilon_{\alpha\beta\gamma\delta} w^\beta w^\gamma w^\delta$$

is the fact that its construction incorporates the components  $\{N e_\alpha\}$  of the vector

$$Nu^\alpha e_\alpha = J$$

Question: What physical properties are expressed by this mathematical tensor of rank (1)?

The answer to this question is as follows:

One takes  $\vec{A}, \vec{B}, \vec{C}$  to be vectors, i.e. displacements, in Minkowski space-time

(no gravitation!). Thus  ${}^*J$  is to be viewed as a tri-linear map on space-time

18-16 ~~#~~  
~~#~~

Thus one has the determinantal map  ${}^*J(\vec{A}, \vec{B}, \vec{C})$  whose vectorial arguments can be spacelike, timelike, or a combination of both.

We shall take

(a)  $(\vec{A}, \vec{B}, \vec{C})$  to be all spacelike, and

(b)  $(\vec{A}, \vec{B}, \vec{C})$   $\vec{A}$  time and  $\vec{B}, \vec{C}$  spacelike.

In either case the value of the determinantal value  ${}^*J(\vec{A}, \vec{B}, \vec{C})$  is given by the formula

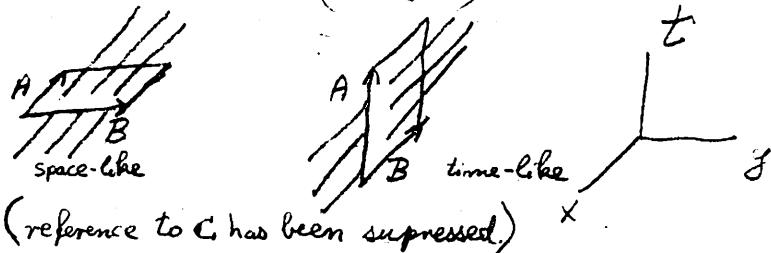
$$\left| \begin{array}{cccc} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \langle \omega^0 | A \rangle & \langle \omega^1 | A \rangle & \langle \omega^2 | A \rangle & \langle \omega^3 | A \rangle \\ \langle \omega^0 | B \rangle & \langle \omega^1 | B \rangle & \langle \omega^2 | B \rangle & \langle \omega^3 | B \rangle \\ \langle \omega^0 | C \rangle & \langle \omega^1 | C \rangle & \langle \omega^2 | C \rangle & \langle \omega^3 | C \rangle \end{array} \right|^{\frac{1}{2}} = {}^*J(A, B, C)$$

If we let (i)  $U : \{u^0, u^1, u^2, u^3\}$  the 4-velocity of a fluid, (ii)  $N$  the particle density in the comoving ("proper") frame, i.e. the frame relative to which  $\{e_1^\mu\} = \{1, 0, 0, 0\}$ ,

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and then consider a three-dimensional parallelopiped spanned by the 4-vectors  $A, B$  and  $C$ , we find that

${}^*J(A, B, C) = \# \text{ of particle world lines}$   
that start or terminate  
in  $(A, B, C)$ .



Expansion of the determinant yields

$${}^*J = N \frac{1}{3!} u^\alpha \epsilon_{\alpha\beta\gamma\delta} \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta$$

= # of particles per (as-yet-unspecified)  
3-volume.

In passing we note that  $J = J^\alpha e_\alpha = Nu^\alpha e_\alpha = Nu$

is the current 4-vector

what does  ${}^*J$  express physically?

18-18 ~~14,9~~  
~~14,14~~

a) If the vectors  $A, B, C$  are spacelike,  
for example

$$\langle \omega^i, A \rangle = \langle \omega^i, B \rangle = \langle \omega^i, C \rangle = 0$$

$$\text{then } {}^*J(A, B, C) = \begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ 0 & \omega^1 B & \omega^2 B & \omega^3 B \\ 0 & \omega^1 C & \omega^2 C & \omega^3 C \end{vmatrix}$$

$$\# \text{ particles} = {}^*J(A, B, C) = Nu^0 \cdot \text{Volume}$$

$$\text{This implies } Nu^0 = \frac{\# \text{ particles}}{\text{volume}}$$

b) If one of the vectors, say  $A$ , is time like  
for example

$$\langle \omega^i, A \rangle = 0 \quad i = 1, 2, 3 \quad \text{but } \langle \omega^0, A \rangle \neq 0$$

while  $B$  and  $C$  are spacelike,

$$\text{then } = \begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \omega^0 A & 0 & 0 & 0 \\ 0 & \langle \omega^1 B \rangle \langle \omega^2 B \rangle \langle \omega^3 B \rangle & 0 & 0 \\ 0 & \langle \omega^1 C \rangle \langle \omega^2 C \rangle \langle \omega^3 C \rangle & 0 & 0 \end{vmatrix} = \begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ 0 & 0 & 0 & 0 \\ 0 & \langle \omega^1 B \rangle \langle \omega^2 B \rangle \langle \omega^3 B \rangle & 0 & 0 \\ 0 & \langle \omega^1 C \rangle \langle \omega^2 C \rangle \langle \omega^3 C \rangle & 0 & 0 \end{vmatrix}$$

$$\# \text{ particles} = {}^*J(A, B, C) = (-)Nu^1 \langle \omega^0, A \rangle \det \begin{vmatrix} \langle \omega^2, B \rangle \langle \omega^3, B \rangle \\ \langle \omega^2, C \rangle \langle \omega^3, C \rangle \end{vmatrix}$$

$$= -Nu^1 \quad (\text{time}) \quad (\text{area } B \times C)_x$$

$$= Nu^1 \quad \text{time} \quad (\text{area } C \times B)_x$$

$$\text{This implies } Nu^1 = \frac{\# \text{ particles}}{(\text{time})(\text{area})} = \text{1st component of particle current}$$

c) Summary:  $\mathcal{J}^0 = Nu^0 = \text{particle density}$   
 $\mathcal{J}^i = Nu^i = \text{particle current density} \quad \left\{ \begin{array}{l} \text{density} \\ \text{particle} \\ \text{4-current} \end{array} \right.$

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14-15

### 3.5 How Construct New Tensors

We already know how to construct new tensors. For example, tensors having the same rank belong to the same tensor space. Hence one can take linear combinations to form new tensors having the same rank. Another example consists of taking tensor products.

For example, given  $\underline{w}^i$  and  $\underline{o}^j$ ,

$$A_{ij} \underline{w}^i \otimes \underline{o}^j$$

is a tensor of rank higher than either  $\underline{w}^i$  and  $\underline{o}^j$ .

There are other ways of producing tensors with different rank.

### 3.4 Tensor Space

Tensors of rank  $(n)$  can be added and multiplied by scalars. More precisely, one has (This is implicit in examples (1-4))  
Proposition: (Tensor Space)

Tensors of rank  $(n)$  form a vector space. This vector space of tensors is denoted by the tensor space  $\underbrace{V \otimes \dots \otimes V}_{n \text{ factors}} \underbrace{V^* \otimes \dots \otimes V^*}_{m \text{ factors}}$

18-21

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20-21~~

### 3.5.1 Raising and Lowering Indices.

Given a metric  $g: V \rightarrow V^*$  or  $(^0)_{ij} \rightarrow (^1)_{ij}$  one has

one has  $u = u^i e_i \text{ and } u^i = g^{ij} u_j$

where  $u_j = g_{ji} u^i$ . (Also,  $\bar{g}: V^* \rightarrow V$ )

$\{u_i\}$  vs  $\{u^i g^{ij} u_j\}$

This correspondence between the components of  $u_i$  and  $u^i$  is called lowering the indices. This correspondence can be generalized to tensors as follows:

Proposition (Lowering of indices)

"lowers" the indices of a tensor; i.e., for example

$$g: \underbrace{V \otimes \dots \otimes V}_{n} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{m} \rightarrow$$

$$\underbrace{V \otimes \dots \otimes V}_{n-1 \text{ factors}} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{m+1 \text{ factors}}$$

$$g: \binom{n}{m} \text{ tensors} \xrightarrow{g} \binom{n-1}{m+1} \text{ tensors}$$

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20-22~~

Explicitly one has

$$H = H^{j_1 \dots j_n}_{i_1 \dots i_m} e_{j_1} \otimes \dots \otimes e_{j_n} \otimes e^{i_1} \otimes \dots \otimes e^{i_m}$$

$$\text{and } H^{j_1 \dots j_n}_{i_1 \dots i_m} e_{j_1} \otimes \dots \otimes e_{j_n} \otimes e^{i_1} \otimes \dots \otimes e^{i_m} = w^{i_1 \dots i_m}$$

or in terms of coordinate components

$$\{H^{j_1 \dots j_n}_{i_1 \dots i_m}\} \text{ and } \{H^{j_1 \dots j_n}_{i_1 \dots i_m}\} =$$

$$= H^{j_1 \dots j_n k}_{i_1 \dots i_m} g_{k j_n}$$

### 3.5.2 Contraction of a Tensor

Regardless of what the metric on  $V$  is, one can lower the rank of a tensor by the contraction map as follows:

Definition (Contraction of a Tensor)

The contraction operation ("map")  $C$  is defined by

$$C: \left\{ \binom{n}{m} \text{ tensors} \right\} \rightarrow \left\{ \binom{n-1}{m-1} \text{ tensors} \right\}$$

$$\{H^{j_1 \dots j_n}_{i_1 \dots i_m}\} \text{ and } \{H^{j_1 \dots j_n k}_{i_1 \dots i_m k}\}$$