Lecture 19

Euclidean group vs
Lorentzian group vs
Symplectic group of transformations
E.V. Arnold §427

on a vector space.
The phase space of the Hamiltonian dynamics of a system is endowed with an antisymmetric ("symplectic") inner product, an antisymmetric tensor of rank (2). A space with such a tensor is said to have a symplectic structure.

Let $V = \mathbb{R}^{2m}$ be an even-dimensional vector space with a covector basis

$$\{ \omega^1 = p_1, \omega^2 = p_2, \ldots, \omega^m = p_m, \omega^{m+1} = q_1, \ldots, \omega^{2m} = q_m \}$$

and its dual basis

$$\{ e_1 = \partial_{p_1}, \ldots, e_m = \partial_{p_m}, e_{m+1} = \partial_{q_1}, \ldots, e_{2m} = \partial_{q_m} \}.$$

Using the fact that

$$\Theta = p_1 \otimes q_1 - q_1 \otimes p_1 \quad (\in V \otimes V^*) \quad (2)$$

one obtains

$$\Theta(e_i, e_j) = \delta_{ij} \delta_{ij} - \delta_{ij} \delta_{ij}.$$

Using Arnold's notation

$$\Theta(\vec{s}, \vec{q}) = \Theta(s_i e_i + s_m e_m, q_i e_i + q_m e_m).$$

Let us compare this symplectic inner product with the familiar inner product.

To do this, let us introduce matrix notation.

$$\Theta(\vec{s}, \vec{q}) = \begin{pmatrix} s_1 & s_2 & \cdots & s_m & s_{m+1} & \cdots & s_{2m} \\ \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \\ s_{m+1} \\ \vdots \\ s_{2m} \\ \end{pmatrix}.$$
or more briefly
\[ \Theta(x, y) = [x^T E y] \]

The form of this inner product corresponds to what in Euclidean space is the choice of an orthonormal or Euclidean basis

\[ g(x, y) = (x^T, y^T)' [1 \ 0 \ 0 \ 1 \ 0 \ \cdots \ 0 \ 1 \ y_n] \]

\[ g = g_{ij} \delta_{ij} \]

i.e. \[ e_i \cdot e_j = \delta_{ij} \] (Euclidean basis)

The symplectic metric in our example has also been chosen relative to a very special basis. In fact, we define a basis \( \{e_i, e_j\} \) as symplectic if

\[ [e_i, e_j] = [e_j, e_i] = 0 \]

\[ (e_i, e_j) = [e_i, e_j] = \delta_{ij} \]

\[ \Theta(e_i, e_j) = -\delta_{ij} \]

Relative to a "symplectically oblique" basis the corresponding inner products would not be so simple, but one would still have a non-degenerate antisymmetric 2-form for which

\[ \Theta(x, y) = -\Theta(y, x) \] (always!)

Comment 1:

What is the property of a linear transformation which takes a Euclidean basis into another Euclidean basis, \( \omega^k = \lambda^k \omega^\ell \)?

The answer is determined by

\[ \omega_i^j = \lambda^k \omega^l \delta_i^k \delta_l^j \]

or in matrix form

\[ \omega^T \Lambda^T \sigma \Lambda \omega = \omega^T \sigma \omega \]

or

\[ \Lambda^T \sigma \Lambda = \sigma \]

Euclidean orthonormal condition for a rotation
The corresponding condition for a Lorentz rotation is
\[
\Lambda^T \eta \Lambda = \eta
\]
where \( \eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). This is the Lorentz O.N. condition for a Lorentz rotation.

A symplectic transformation is characterized by
\[
\Lambda^T E \Lambda = E
\]
where \( E = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \). This is the symplectic condition for a linear canonical transformation.

Given two transformations, say \( \Lambda_1 \) and \( \Lambda_2 \), then \( \Lambda_1, \Lambda_2 \) and \( \Lambda_1^T \Lambda_2 \) satisfy the same constraints:
\[
(\Lambda_1 \Lambda_2)^T \eta (\Lambda_1 \Lambda_2) = \Lambda_1^T \Lambda_2^T \eta \Lambda_1 \Lambda_2 = \Lambda_1^T \eta \Lambda_2 = \eta
\]
\[
(\Lambda^T)^T \eta \Lambda^T = (\Lambda^T)^{-1} \Lambda^T \eta \Lambda^T = \eta
\]

A set of transformations which (a) is closed under multiplication, (b) has an identity element, (c) has an inverse for every transformation, is called a transformation group. The matrices \( \Lambda, \eta \) and \( E \) characterize

\( O(n) \): orthogonal group, the group of orthogonal transformations.

\( O(3) \): Lorentz group, the group of Lorentz transformations.

\( Sp(n) \): symplectic group, the group of symplectic transformations.

For more on these, see C. Chevalley: Theory of Lie Groups.