

Lecture 19

Euclidean group vs

Lorentzian group vs

Symplectic group of transformations
[V. Arnold §42]

on a vector space.

Fri Oct 7

I.

Q: What kind of transformation takes an o.n. (Euclidean/Lorentz) basis into another basis which is also o.n. ?

$$\begin{aligned}
 A: \vec{x} \cdot \vec{y} &= x^k e_k \cdot e_l y^l = g_{kl} \omega^k \otimes \omega^l(x, y) \\
 &= \lambda^k_i x^i e_k \cdot e_l \lambda^l_j y^j \\
 &= \lambda^k_i \omega^i(x) g_{kl} \lambda^l_j \omega^j(y) \\
 &= \lambda^k_i g_{kl} \lambda^l_j \omega^i \otimes \omega^j(x, y) \\
 &= (\Lambda^T)_i^k g_{kl} \lambda^l_j \omega^i \otimes \omega^j(x, y) (*) \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 x \cdot y &= x^i g_{ij} y^j \\
 &= g_{ij} \omega^i(x) \omega^j(y)
 \end{aligned}$$

$$x \cdot y = g_{ij} \omega^i \otimes \omega^j(x, y) \quad (**)$$

CONCLUSION:

Equality holds $\forall x, y$

$$(*) \& (**) \Rightarrow g_{ij} = (\Lambda^T)_i^k g_{kl} \lambda^l_j \quad \begin{matrix} \text{(index)} \\ \text{(notation)} \end{matrix}$$

Equivalently one has

$$G = \Lambda^T G \Lambda \quad \begin{matrix} \text{(matrix)} \\ \text{(notation)} \end{matrix}$$

II. Application of $G = \lambda^T G \lambda$ to the -19,2-
relation between a Jacobian determinant
to the determinant of the metric rep've:

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \gamma_{\mu\nu} dx^\mu \otimes dx^\nu$$

$$\text{Let } [J^{\mu}_{\nu}] = [\frac{\partial x^\mu}{\partial x^{\nu}}] = J$$

be the Jacobian matrix from
 $\{x^\mu\}$ to $\{x^\nu\}$

Then

$$g_{\mu\nu} = \frac{\partial x^\mu}{\partial x^\nu} \gamma_{\mu\nu} \frac{\partial x^\nu}{\partial x^\mu}$$

$$G' = J^T \gamma J$$

$$\therefore \det g_{ij} = \det J^T \underbrace{\det \gamma}_{\det J^{-1}} \det J$$

$$\text{CONCLUSION: } = -(\det J)^2$$

$$\therefore \boxed{\det J = \sqrt{-\det g}}$$

Application!

$$\begin{aligned} \therefore E &= E_{i_1 \dots i_m} dx^{i_1} \otimes \dots \otimes dx^{i_m} \\ &= E_{i_1 \dots i_m} dx^{i_1} \wedge \dots \wedge dx^{i_m} \quad i_1 < \dots < i_m \\ &= E_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} \end{aligned}$$

$$E_{i_1 \dots i_n} = [i_1 \dots i_n] J = [i_1 \dots i_n] \sqrt{-g}$$

III. Examples of $G = \Lambda^T \Gamma \Lambda$

Example 3

1. Euclidean metric

$$g_{ij} = \delta_{ij}$$

$$\boxed{\delta = \Lambda^T \Gamma \Lambda} \quad \text{or } I = \Lambda^T \Lambda$$

$$g_{k'l'} = \delta_{k'l'}$$

$$\text{where } \delta = I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

2. Minkowski metric

$$g_{\mu\nu} = \gamma_{\mu\nu}$$

$$\boxed{\gamma = \Lambda^T \eta \Lambda}$$

$$g_{\mu' \nu'} = \gamma_{\mu' \nu'}$$

$$\text{where } \gamma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comment: 9) The set of matrices satisfying

$$\gamma = \Lambda^T \eta \Lambda \quad \text{form a group.}$$

$$\text{Indeed } \gamma = \Lambda_1^T \eta \Lambda_1 \quad \gamma = \Lambda_2^T \eta \Lambda_2 \Rightarrow \gamma = (\Lambda_1 \Lambda_2)^T \eta (\Lambda_1 \Lambda_2)$$

Furthermore,

$$(\Lambda^{-1})^T \gamma \Lambda^{-1} = (\Lambda^{-1})^T \Lambda^T \eta \Lambda \Lambda^{-1} \Rightarrow \gamma = (\Lambda^{-1} \Lambda^{-1})^T \eta (\Lambda \Lambda^{-1}) = \eta \Rightarrow (\Lambda^{-1})^T \gamma \Lambda^{-1} = \eta$$

Hence we have closure under group multiplication and existence of an inverse for each λ .

This group is called the Lorentz group and is designated by $O(1, 3)$

b) The group of Euclidean rotations is characterized by elements λ that satisfy $[s = \lambda^T s \lambda]$

The Euclidean group in n dimensions is designated by $O(n)$

3. Symplectic metric

A vector space \mathbb{R}^n is said to have a Euclidean structure if that space is mathematized by a Euclidean scalar product.

Such an scalar product is defined by a symmetric tensor of rank (2), whose constitutive properties (i.e. those which are not mentioned in the definition but nevertheless are part of the concept) include the fact that it admits an orthonormal (o.n.) basis for \mathbb{R}^n .

The Euclidean structure on R^n is characterized by a symmetric scalar product that manifests itself by the existence of an appropriately chosen basis which is $0, n.$

In the same way the Symplectic structure on R^{2n} is characterized by an antisymmetric scalar product that manifests itself by the existence of an appropriately chosen basis for $R^{2n}.$

The phase space of the Hamiltonian dynamics of a system (CONT'D ON NEXT PAGE)

(i.e. any system governed by a set of dynamical equations having the form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \text{ and } \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

is mathematized by an antisymmetric ("symplectic") scalar product, i.e., an antisymmetric tensor of rank $\binom{n}{2}$.

A space with such a tensor field is said to have a symplectic structure.

Indeed, let $V = \mathbb{R}^{2n}$ be an even-dimensional vector space with a covector basis (for V^*)

$$\{\omega^1 = dp_1, \omega^2 = dp_2, \dots, \omega^n = dp_n; \omega^{n+1} = dq^1, \omega^{n+2} = dq^2, \dots, \omega^{2n} = dq^n\}$$

and the corresponding dual basis

$$\{e_{p_1}, e_{p_2}, \dots, e_{p_n}; e_{q_1}, e_{q_2}, \dots, e_{q_m}\}$$

so that

$$\langle dp_k | e_{p_l} \rangle = \delta_{kl}$$

$$\langle dp_k | e_{q_i} \rangle = 0$$

$$\langle dq^k | e_{p_i} \rangle = 0$$

$$\langle dq^k | e_{q_l} \rangle = \delta_{kl}$$

Using this basis, consider the anti-symmetric tensor of rank $\binom{0}{2}$

$$\Theta = dp_1 \wedge dq^1 + dp_2 \wedge dq^2 + \dots + dp_n \wedge dq^n = dp_k \wedge dq^k.$$

The symplectic scalar product of two vectors

(CONT'D ON NEXT PAGE)

$$\vec{\xi} = \xi_i e_{p_i} + \xi_{n+i} e_{q_i}$$

$$\vec{\eta} = \eta_j e_{p_j} + \eta_{n+j} e_{q_j}$$

$$\Theta(\vec{\xi}, \vec{\eta}) = [\vec{\xi}, \vec{\eta}] \quad (\text{Arnold's notation})$$

$$= \Theta(\xi_i e_{p_i} + \xi_{n+i} e_{q_i}, \eta_j e_{p_j} + \eta_{n+j} e_{q_j})$$

Using the fact that

$$\Theta = p_k \otimes q_k - q_k \otimes p_k \quad (\in V^* \otimes V^*) \quad (1)$$

$$\Theta(e_{p_i}, e_{p_j}) = 0 = \Theta(e_{q_i}, e_{q_j}), \quad \Theta(e_{p_i}, e_{q_j}) = \delta_{ki} \delta_{kj} = \delta_{ij}$$

one obtains

$$\Theta(\vec{\xi}, \vec{\eta}) = \xi_i \delta_{ki} \eta_{n+j} \delta_{kj} - \xi_{n+i} \delta_{ki} \eta_j \delta_{kj}$$

$$= \xi_k \eta_{n+k} - \xi_{n+k} \eta_k$$

Let us compare this symplectic inner product with the familiar inner product.

To do this, let us introduce matrix notation.

$$\Theta(\vec{\xi}, \vec{\eta}) = (\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_{2n}) \begin{bmatrix} & & & & \\ & 0 & 1 & 0 & \\ & 1 & 0 & \ddots & \\ & \vdots & \vdots & \ddots & 1 \\ & -1 & -1 & \ddots & \\ & 0 & 0 & \ddots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \ddots & 0 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_n \\ \eta_{n+1} \\ \vdots \\ \eta_{2n} \end{pmatrix}$$

components
of Θ relative to
the given basis

or more briefly

$$\Theta(\vec{e}, \vec{y}) = [\vec{e}] [E] [\vec{y}]$$

The form of this inner product corresponds to what in Euclidean space is the choice of an orthonormal or Euclidean basis

$$g(\vec{x}, \vec{y}) = (x^1 \dots x^n) \begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = x^T \delta y$$

$g = g_{ij} \omega \otimes \omega^i$

$$\text{i.e. } e_i \cdot e_j = \delta_{ij} \quad (\text{Euclidean basis})$$

The symplectic metric in our example has also been chosen relative to a ^(symplectically o.n.) very special basis. In fact we DEFINE a basis $\{e_{p_i}, e_{q_j}\}$ as symplectic if

$$[e_{p_i}, e_{p_j}] = [e_{q_i}, e_{q_j}] = 0$$

$$\Theta(e_{p_i}, e_{q_j}) = [e_{p_i}, e_{q_j}] = \delta_{ij}$$

$$\Theta(e_{q_i}, e_{p_j}) = [e_{q_i}, e_{p_j}] = -\delta_{ij}$$

$$\Theta = p_k \otimes q_k - q_k \otimes p_k$$

relative to
a symplec
basis relative
to which

$$\Theta(\vec{e}, \vec{y}) =$$

$$= (\vec{e}_1 \dots \vec{e}_{2n}) \begin{bmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix}$$

A symplectic transformation Λ :

$$\begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_n \\ \xi'_{n+1} \\ \vdots \\ \xi'_{2n} \end{pmatrix} = \Lambda \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \\ \xi_{n+1} \\ \vdots \\ \xi_{2n} \end{pmatrix}$$

is one leaves

$$E = \begin{bmatrix} \cdot & 0 & 0 \\ 0 & \ddots & 0 \\ \vdots & 0 & 0 \end{bmatrix}$$

on page 19.8 invariant, i.e.

$$\boxed{\Lambda^T E \Lambda = E} \quad (*)$$

This is the symplectic condition for
a (linear) canonical transformation.

Comment: The set of symplectic transformations Λ which satisfy $\Lambda^T E \Lambda = E$ form a group, the symplectic group $Sp(n)$.

Summary

There are three fundamental groups which mathematize the invariance of the laws of the physical world:

Rotation group $O(n)$: $\Lambda^T \delta \Lambda = \delta$

Lorentz group $O(1,3)$: $\Lambda^T \gamma \Lambda = \gamma$

Symplectic group $Sp(n)$: $\Lambda^T E \Lambda = E$