

Lecture 20

Manifolds

- a) Where do they come from?
- b) A conceptual overview

Coordinate charts, atlas, manifold

{Coordinate representative of a function.

[MTW Ch. 9; Hicks P1-4; Singer & Thorpe P97-99]

20.2

20.1

MANIFOLDS (Where do they come from?)

As the name implies, non-linear mathematics can be understood only after one has grasped linear mathematics. The driving force behind non-linear mathematics comes from astronomy, classical mechanics, curved surfaces, dynamical systems, and many other realms of the mathematical universe, ranging from the subatomic through the biological, to the cosmic.

But all of them, as Newton showed us (via his method of infinitimals) depend on linear mathematics as a precondition

for understanding them

The central concept of linear mathematics is that of a vector space, one of finite dimensions in particular. The central concept of non-linear mathematics, on the other hand, is that of a manifold.

As we shall see, the concept of a manifold is based on a generalization of two key properties of a vector space, but a generalization which does not depend on their linearity ("closure under linear combinations"). The first property is the concept of bases-induced coordinate representations and the transition transformations between them.

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The second one is the concept of real valued linear functions defined on the vector space.

I) Basis-induced Coordinate Representations
and their
Transition Matrices.

(Go to page 20.4)

I.) Bases-induced Coordinate Representations
and their

^{20.4}

Transition Maps (Matrices)

Let $\{\mathbf{e}_i\}_{i=1}^n = B$ and $\{\bar{\mathbf{e}}_i\}_{i=1}^n = C$ be two bases

for the vector space V , and let $\{\omega^i\}_{i=1}^n = B^*$ and

$\{\bar{\omega}^i\}_{i=1}^n = C^*$ be the corresponding sets

("dual bases") of coordinate functions.

Each of these bases induces its 1-1

linear transformations from V to copies

of \mathbb{R}^n

$$\varphi_B = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix}$$

$$\bar{\varphi}_C = \begin{bmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^n \end{bmatrix}$$

$$V \xrightarrow{\varphi_B: R^n} R^n \quad P_{CB} = \bar{\varphi}_C \circ \varphi_B^{-1}$$

= "transition
matrix"

$$V \xrightarrow{\varphi_B: R^n} R^n \quad P_{CB} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad P_{CB} = \bar{\varphi}_C \circ \varphi_B^{-1}$$

$$V \xrightarrow{\bar{\varphi}_C: R^n} R^n \quad \bar{\varphi}_C = \begin{bmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{bmatrix} \quad P_{BC} = \varphi_B \circ \bar{\varphi}_C^{-1}$$

Comment 1:

Notice that both φ_B and $\bar{\varphi}_C$ are isomorphism between V and copies of \mathbb{R}^n .

These copies are coordinate representations of V . Indeed one has

$$\varphi_B = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} \text{ and } \varphi_B^{-1} = [\mathbf{e}_1, \dots, \mathbf{e}_n]$$

$$\bar{\varphi}_C = \begin{bmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^n \end{bmatrix} \text{ and } \bar{\varphi}_C^{-1} = [\bar{\mathbf{e}}_1, \dots, \bar{\mathbf{e}}_n]$$

so that

$$\varphi_B \circ \varphi_B^{-1} = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} [\mathbf{e}_1, \dots, \mathbf{e}_n] = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I$$

is the identity on \mathbb{R}^n . Similarly

$$\bar{\varphi}_C \circ \bar{\varphi}_C^{-1} = I$$

Comment 2: If $\vec{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in \mathbb{R}^n$

$$\text{then } \bar{\varphi}_C(\vec{v}) = \begin{bmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{bmatrix}$$

$$\bar{\varphi}_C(\vec{v}) = [\bar{\mathbf{e}}_1, \dots, \bar{\mathbf{e}}_n] \begin{bmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{bmatrix} = \bar{\mathbf{e}}_1 \cdot \vec{v} = \vec{v} \in V,$$

as it must be.

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Comment 3:

Also notice that there exist transition maps between the copies of R^n , namely

$$P_{CB} = \bar{\varphi}_C \circ \bar{\varphi}_B^{-1}$$

and

$$P_{BC} = \varphi_B \circ \bar{\varphi}_C^{-1} (= P_{CB}^{-1})$$

Each of these composite maps is a matrix from one copy of R^n to another copy of R^n
 $R^n : P_{CB} \rightarrow R^n$

The entries P_{ij}^T of the matrix

$$P_{CB} = [P_{ij}^T]$$

are determined by the fact that for every vector

$$\vec{v} = e_i \cdot v^i = \bar{e}_j \cdot \bar{v}^j$$

one must have

$$\begin{bmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{bmatrix} = P_{CB} \cdot \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

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Nota bene:

The index notation, in terms of which one

has

$$\{v^i\} \mapsto \bar{v}^j = P^T_j \cdot v^i \quad (j = 1, \dots, n),$$

is more powerful in that it determines (i.e. tells us how to calculate) the columns of the transition matrix $[P_{ij}^T]$

Indeed, by expanding the vector \vec{v} in terms of the old basis $\{e_i\}$ and also in terms of the new basis $\{\bar{e}_j\}$ one has

$$e_i \cdot v^i = \vec{v} = \bar{e}_j \cdot \bar{v}^j = \bar{e}_j \cdot P^T_j \cdot v^i$$

or

$$\begin{bmatrix} e_1 & \dots & e_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \vec{v} = \begin{bmatrix} \bar{e}_1 & \dots & \bar{e}_n \end{bmatrix} \begin{bmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{bmatrix} =$$

$$= \begin{bmatrix} \bar{e}_1 & \dots & \bar{e}_n \end{bmatrix} \begin{bmatrix} P_1 & & & & v^1 \\ & \ddots & & & \vdots \\ & & P_n & & v^n \end{bmatrix}$$

20.8

$$[e_1, \dots, e_m] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \vec{v} = [\bar{e}_1 P^{\ddagger}, \dots, \bar{e}_m P^{\ddagger}] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

This equation holds for all column arrays.

Thus one has

$$e_i = \bar{e}_j P^{\ddagger} \text{ etc.} \quad (\text{the transition matrix})$$

In other words, the columns of $P_{CB} = [P^{\ddagger}_i]$

are the expansion coefficients of the old (B) in terms of the new (C) basis vectors.

To summarize,

given two coordinate representations of (the elements of) a vector space, there is an isomorphism between these representations,

In linear algebra this isomorphism is

called a transition matrix, and it is

given by

$$P_{CB} = \bar{\phi}_c \circ \bar{\phi}_B^{-1} = \begin{bmatrix} \bar{w}^1 \\ \bar{w}^2 \\ \vdots \\ \bar{w}^n \end{bmatrix} [e_1, e_2, \dots, e_m]$$

$$= \begin{bmatrix} \bar{w}^1(e_1) & \bar{w}^1(e_2) & \dots & \bar{w}^1(e_n) \\ \bar{w}^2(e_1) & \bar{w}^2(e_2) & \dots & \bar{w}^2(e_n) \\ \vdots & \vdots & & \vdots \\ \bar{w}^n(e_1) & \bar{w}^n(e_2) & \dots & \bar{w}^n(e_n) \end{bmatrix}$$

In other words, as we have already seen near the top of the previous page, the columns of the

$$\text{transition matrix } P_{CB} = \bar{\phi}_c \circ \bar{\phi}_B^{-1} = [P^{\ddagger}_i]$$

are the expansion coefficients of the old basis vectors $\{e_i\} = B$ in terms of the new basis

vectors $\{\bar{e}_j\} = C$.

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20.10

II.) Real Valued Linear Functions
Defined on a Vector Space

Consider a linear function $\underline{f} \in V^*$:

$$\underline{f}: V \longrightarrow \mathbb{R}$$

For each basis-induced coordinate system,

say $\underline{\varphi}_B = \begin{bmatrix} \underline{\omega}^1 \\ \vdots \\ \underline{\omega}^n \end{bmatrix}$ $B = \{\underline{e}_1, \dots, \underline{e}_m\}$

and

$$\underline{\varphi}_C = \begin{bmatrix} \underline{\omega}'^1 \\ \vdots \\ \underline{\omega}'^m \end{bmatrix} \quad C = \{\underline{e}'_1, \dots, \underline{e}'_m\}$$

there exist corresponding coordinate representatives of \underline{f} , namely

$$[f_1, \dots, f_n] \quad \text{where } f_i = \underline{f}(\underline{e}_i)$$

and

$$[f'_1, \dots, f'_n] \quad \text{where } f'_j = \underline{f}(\underline{e}'_j)$$

so that

$$\underline{f} \cdot \underline{\omega}'^i = \underline{f} = \underline{f}'_j \underline{\omega}'^j \in V$$

20.11

Note that

The components of these coordinate representatives of \underline{f} are the components of the composite map $\underline{f} \circ \underline{\varphi}_B^{-1}$. Indeed, one has

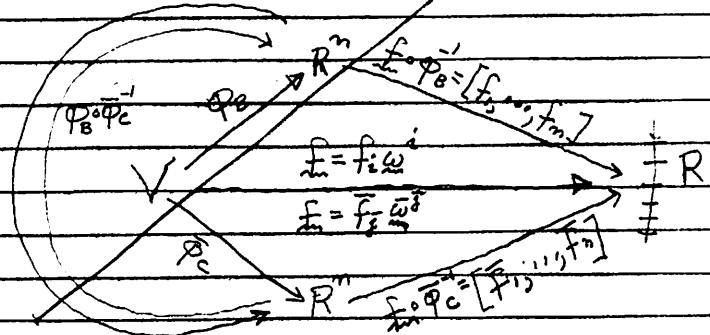
$$\begin{aligned} \underline{f} \circ \underline{\varphi}_B^{-1} &= \underline{f}([e_1, \dots, e_m]) \\ &= [f_1(e_1), \dots, f_n(e_m)] \\ &= [f_1, \dots, f_n], \end{aligned}$$

which coincides with the result on the previous page 20.8, as it must!

Putting everything together one has the following constellation of concepts

Do this later 20.11
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$\underline{\varphi}_C \circ \underline{\varphi}_B^{-1}$



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Furthermore, by evaluating $\underline{f} \circ \bar{\varphi}_c^{-1}$

at a particular point $(v^1, \dots, v^n) = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$ in R ,

one obtains the value

$$\underline{f} \circ \bar{\varphi}_c^{-1}(v^1, \dots, v^n) = \underline{f} \circ \bar{\varphi}_c^{-1}\left(\begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}\right)$$

$$= [f_1, \dots, f_n] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

$$= \underline{f}_c v^i$$

Thus one must call $\underline{f} \circ \bar{\varphi}_c^{-1}$ the b-coordinate representative of \underline{f} . On the other hand

$\underline{f} \circ \bar{\varphi}_b^{-1}$ is called the c-coordinate representative of \underline{f} . One can use either

representative to evaluate $\underline{f}(\vec{v})$:

$$\underline{f}(\vec{v}) = \underline{f} \circ \bar{\varphi}_b^{-1}(v^1, \dots, v^n) = \underline{f}_b v^i$$

$$\underline{f}(\vec{v}) = \underline{f} \circ \bar{\varphi}_c^{-1}(\bar{v}^1, \dots, \bar{v}^n) = \underline{f}_c \bar{v}^i$$

20.13

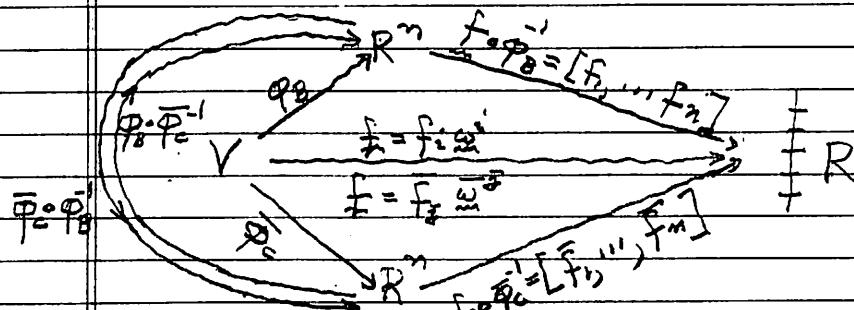
The obtained values are the same, as they must!

We shall see that the concept "coordinate representative" of a (scalar-valued) function carries over to the conceptual framework surrounding a manifold. However, the linearity

(i.e. the "super-position principle") does not.

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Putting every together, one has the following framework of concept



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We shall now see that the formation of the concept of a manifold is based on the framework of concepts, mappings and their domains, exhibited on pages 20.4 and 20.11. However, linearity, vector addition, and superposition will not be part of the manifold concept. Instead, there will be a new feature, namely differentiability, which will be introduced explicitly.

Manifolds (An overview)

A manifold is a topological space which locally (to a "near sighted observer") looks like a rectilinear space with a Cartesian coordinate system.

At each point of a differentiable manifold there is a tangent space which is the vector space of all tangent vectors. These tangent vectors are the directional derivatives that operate on differentiable real valued functions whose domain is the manifold. The set of partial derivatives along the coordinate lines of a coordinate system form a basis of the tangent space. The set of differentials of the coordinate functions form a basis for the corresponding (dual) cotangent space.

- 1.) a. Manifold 20.16
- b. Coordinate system
- 2.) tangent vector = directional derivative
- 3.) a. partial derivatives = basis vectors
b. diff's = elements of dual space of covectors

Ch. 9 & 8 in MTW

I.M. Singer & Thorpe: Lecture Notes on Topology &

20.17

Diff'l Geometry p97-99

J.Hicks: Notes on Diff'l Geometry Pl-4

Manifolds

G. Sorani: An Introduction to real & complex

Definition of a Manifold

Manifolds QA611

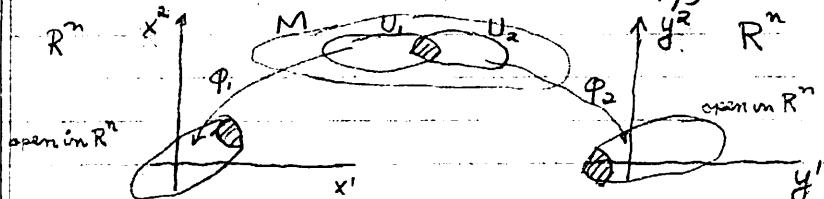
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Def A. Coordinate Charts Read Misner handout.
P 890-895

(1) Consider a set M

(2) (a) A local ("coordinate") chart is a mapping (or homeomorphism) φ from an nbhd $U \subset M$ into cartesian space \mathbb{R}^n , together with U

(b) Such a chart is written as (φ, U)



(c) Two overlapping charts (φ_1, U_1) & (φ_2, U_2) are C^r related \Leftrightarrow

$(x^1, \dots, x^n) = \varphi_1 \circ \varphi_2^{-1}(y^1, \dots, y^n)$ and $(y^1, \dots, y^n) = \varphi_2 \circ \varphi_1^{-1}(x^1, \dots, x^n)$ are "transition map"

C^r maps, i.e. have continuous partials of order r .

(Note: $\varphi_1 \circ \varphi_2^{-1}$ is called a transition map, See example)

B. Atlas: An n -atlas is a collection of C^r -related charts $\{(\varphi_i, U_i)\mid i=1,2,\dots\}$ such that $M = \bigcup U_i$

C. An n -dimensional C^r -manifold (or a C^r - n -manifold) is a set M together with a "maximal" C^r - n -atlas. A maximal atlas contains all possible charts.