Lecture 20

Manifolds

a) Where do they come from?

b) A conceptual overview

Coordinate charts, atlas, manifold

Coordinate representative of a function.

[MTW Ch. 9; Hicks P1-4; Singer & Thorpe P97-99]
The purpose of mathematics is to mathematize causal relations in the physical world.

In astronomy, physics, classical and quantum mechanics, classical and quantum fields, dynamical systems, neuroscience, biology, curved surfaces, i.e., in the realm of the mathematical universe ranging from the subatomic, thru the biological, to the cosmic.
MANIFOLDS (Where do they come from?)

As the name implies, non-linear mathematics can be understood only after one has grasped linear mathematics. The driving force behind non-linear mathematics comes from astronomy, classical mechanics, curved surfaces, dynamical systems, and many other realms of the mathematical universe, ranging from the subatomic through the biological, to the cosmic.

But all of them, as Newton showed us (via his method of infinitesimals) depend on linear mathematics as a precondition.
for understanding them.

The central concept of linear mathematics is that of a vector space, one of finite dimensions in particular. The central concept of non-linear mathematics, on the other hand, is that of a manifold.

As we shall see, the concept of a manifold is based on a generalization of two key properties of a vector space, but a generalization which does not depend on their linearity ("closure under linear combinations"). The first property is the concept of bases-induced coordinate representations and the transition transformations between them.
The second one is the concept of real valued linear functions defined on the vector space.

I) Basis-induced Coordinate Representations and their Transition Matrices.

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1) Bases-induced Coordinate Representations and their Transition Maps (Matrices)

Let \( \{ e_i \}_{i=1}^n = B \) and \( \{ e_i \}_{i=1}^n = C \) be two bases for the vector space \( V \) and let \( \{ \omega^i \}_{i=1}^n = B^* \) and \( \{ \omega^i \}_{i=1}^n = C^* \) be the corresponding sets ("dual bases") of coordinate functions.

Each of these bases induces \( 1 \)-\( 1 \) linear transformations from \( V \) to copies of \( \mathbb{R}^n \).

\[
B = \{ e_1, \ldots, e_n \} \\
C = \{ \omega_1, \ldots, \omega_n \} \\
P_{B \rightarrow C} = \Phi_c \Phi_b^{-1} \\
\phi_B = \begin{bmatrix} \omega^1 \\ \vdots \\ \omega^n \end{bmatrix} \\
\phi_C = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}
\]
Comment 1:
Notice that $\varphi_B$ and $\bar{\varphi}$ are isomorphism between $V$ and copies of $\mathbb{R}^m$. Each of these copies is a coordinate representation of $V$. Indeed one has

$$
\varphi_B = \left[ \begin{array}{c}
\omega^1 \\
\vdots \\
\omega^m
\end{array} \right]: \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad u \mapsto \left[ \begin{array}{c}
\omega^1(u) \\
\vdots \\
\omega^m(u)
\end{array} \right] = \left[ \begin{array}{c}
\bar{u}^1 \\
\vdots \\
\bar{u}^m
\end{array} \right] \in \mathbb{R}^m \quad \text{(coordinate of } \bar{u} \text{)}
$$

$$
\varphi_B^{-1} = \left[ e_1 \ldots e_n \right]: \mathbb{R}^m \longrightarrow V, \quad u \mapsto \left[ \begin{array}{c}
\bar{u}^1 \\
\bar{u}^m
\end{array} \right] = \left[ \begin{array}{c}
e_1 \cdot \bar{u}^1 \\
\vdots \\
e_n \cdot \bar{u}^m
\end{array} \right] = e_B u^B = \bar{v} \in V
$$

The composites $\varphi_B \circ \varphi_B^{-1}$ and $\varphi_B^{-1} \circ \varphi_B$ of these maps are the identity maps for $\mathbb{R}^m$ and $V$ respectively.

Right inverse: $\varphi_B \circ \varphi_B^{-1} = \left[ \begin{array}{c}
\omega^1 \\
\vdots \\
\omega^m
\end{array} \right] \left[ e_1 \ldots e_n \right] = \left[ \begin{array}{c}
\omega^1(e_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \omega^m(e_m)
\end{array} \right] = \left[ \begin{array}{c}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{array} \right]

= I_{\mathbb{R}^m} \quad \text{(identity matrix on } \mathbb{R}^m \text{)}$
\[ \begin{aligned}
\phi_B^\dagger \circ \phi_B &= \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} \left[ \omega^1 \right] \\
&= e_1 \otimes \omega^1 + e_2 \otimes \omega^2 + \cdots + e_n \otimes \omega^n \\
&= e_i \otimes \omega^i = I \quad (\text{identity tensor map on } V)
\end{aligned} \]

Indeed, letting \( \vec{v} \in V \) one has
\[ \phi_B^\dagger \circ \phi_B (\vec{v}) = e_i \otimes \omega^i (e_k \vec{v}^k) \]
\[ = e_i \delta_i^k \vec{v}^k = e_k \vec{v}^k = \vec{v} \quad \forall \vec{v} \in V \]

Conclusion: The identity map
\[ \phi_B^\dagger \circ \phi_B = e_i \otimes \omega^i = \delta^i_k e_i \otimes \omega^k = dP \]
on \( V \) is a tensor of rank (1).
\[ dP = e_i \otimes \omega^i = (\phi_B^\dagger \circ \phi_B) \]

It is called Cartan's unit tensor.

It mathematizes the concept of "moment" in the Cartan-Wheeler formulation (see §15.4 in MTW) of the Einstein field equations.
As noted before, if \( \vec{v}_B = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \) is the \( B \)-representative of \( \vec{v} \), then

\[
\varphi_B(\vec{v}_B) = [e_1, \ldots, e_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v^k = \vec{v} \in V
\]
as it must be.

Comment 3:

Also notice that there exist transition maps between the copies of \( \mathbb{R}^n \), namely

\[
P_{CB} = \overline{\varphi}_C \circ \overline{\varphi}_B
\]

and

\[
P_{BC} = \varphi_B \circ \overline{\varphi}_C = (P_{CB})^{-1}
\]

Each of these composite maps is a matrix from one copy of \( \mathbb{R}^n \) to another copy of \( \mathbb{R}^n \)

\[
\begin{array}{ccc}
R^n & \xrightarrow{P_{CB}} & R^n
\end{array}
\]

The entries \( P^i_j \) of the matrix

\[
P_{CB} = [P^i_j]
\]

are determined by the fact that for every vector

\[
\vec{v} = e_i \quad v^i = e_j \quad \overline{v}^j
\]

one must have

\[
\begin{bmatrix}
\vec{v}_1 \\
\vdots \\
\vec{v}_n
\end{bmatrix} = P_{CB}
\begin{bmatrix}
v_1 \\
\vdots \\
v_n
\end{bmatrix}
\]
Def A. **Coordinate Charts**

1. Consider a set \( M \)

2. A local ("coordinate") chart is a mapping (or "homeomorphism") \( \varphi \) from a nbhd \( U \subset M \) into cartesian space \( \mathbb{R}^n \), together with \( U \) and \( \varphi \).

3. Such a chart is written as \( (\varphi, U) \).

4. Two overlapping charts \( (\varphi_1, U_1) \) and \( (\varphi_2, U_2) \) are \( C^r \) related if \( (x', \ldots, x^n) = \varphi_1 \circ \varphi_2^{-1}(y', \ldots, y^n) \) and \( (y', \ldots, y^n) = \varphi_2 \circ \varphi_1^{-1}(x', \ldots, x^n) \) are \( C^r \) maps, i.e. have continuous partials of order \( r \).

(Note: \( \varphi_1 \circ \varphi_2^{-1} \) is called a transition map. See example.)

Def B. **Atlas**: An \( n \)-atlas is a collection of \( C^r \)-related charts \( \{(\varphi_i, U_i)\}_{i=1}^\infty \) such that \( M = \bigcup U_i \).

Def C. An \( n \)-dimensional \( C^r \)-manifold (or a \( C^r \)-manifold) is a set \( M \) together with a "maximal" \( C^r \)-atlas. A maximal atlas contains all possible charts.
We shall assume without further discussion:

Bolzano-Weierstrass's Theorem
Every compact set has a finite open covering

Lindelöf’s Theorem
Every separable metric space has a countable open covering