Two basic concepts in Differential Topology.

1. <u>Coordinate</u> <u>charts</u> on the manifold S^2.
2. <u>Vector</u> <u>tangent</u> to a manifold.

① Example of two C^∞ coordinate charts on S^2; ② vector tangent to a manifold
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Euler angles θ, ϕ do not change along this curve, one may label the different loops in S^3 by the position of the 3-axis in Fig. 3, i.e., by a point of S^2. Each point p in S^3 corresponds to a definite rotation $A(p)$, to a definite configuration of the 123 axes in Fig. 3, and in particular to a definite location $q \in S^2 \subset E^3$ for the unit vector along the 3 axis. Thus we have defined a differentiable mapping

$$\pi : S^3 \to S^2 : p \to q = pkp^{-1}. \tag{1.11}$$

For each fixed $q \in E^2$, the set of points of S^3 which map into q, $\pi^{-1}(q)$, is a circle S^1. (This relationship can be described by saying that π represents S^3 as an S^1-bundle over S^2.)

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Thus far we have not introduced any coordinates on the manifolds we have discussed; however, the possibility of covering any sufficiently small region on the manifold by a set of local coordinates is the essential and characteristic feature of differentiable manifolds. I will illustrate the idea of local coordinates on S^2. First, note that it is generally impossible to use a single coordinate system for the whole manifold. The sphere S^2 is frequently described in terms of co-latitude and longitude angles θ and ϕ, so the standard metric on S^2 is written

$$ds^2 = d\theta^2 + sin^2\theta \, d\phi^2 \tag{1.12}$$

This attempt to use a single set of coordinates is convenient, provided the geometry is so familiar we can ignore obvious formal difficulties, such as the singularities

$$g^{\phi\phi} = (\sin\theta)^{-2} \to \infty$$

for

$$\theta \to 0, \pi$$

When we expect the formal mathematics rather than our intuition to carry the burden of deciding questions of differentiability, we must restrict the idea of a coordinate system in such a way that the components of differentiable tensor fields will always be differentiable functions. The first requirement then is that we choose as coordinates only differentiable functions, which neither θ nor ϕ is at the poles $\theta = 0, \pi$.

Since a point $\mathbf{x} \in S^2$ is a triple (x_1, x_2, x_3) of real numbers, we can define three real valued functions by the rules

$$x(\mathbf{x}) = x_1$$
$$y(\mathbf{x}) = x_2$$
$$z(\mathbf{x}) = x_3$$

Because each of these functions is differentiable when one lets $\mathbf{x}$ vary over E^3, it is also for $\mathbf{x} \in S^2$. In suitable regions one can take pairs of these functions as local coordinates on S^2. For instance, let

$$N = \{\mathbf{x} \in S^2 | z(\mathbf{x}) > 0\}$$

Then to any point $\mathbf{x}$ in this northern hemisphere N there corresponds a unique point $(x, y) \in E^2$ by the rule

$$(x, y) \leftrightarrow \mathbf{x} = (x, y, \sqrt{1 - x^2 - y^2}).$$

Thus the pair of functions x, y on N gives a one to one correspondence between N and a region $(x_1^2 + x_2^2 < 1)$ of E^2. To any function f defined on S^2 or N there is a corresponding function f_N defined on this part of E^2 by the rule

$$f_N(x, y) = f(\mathbf{x})$$

The coordinate systems one admits and the differentiable functions on the manifold must always be consistent in the sense that $f(\mathbf{x})$ is differentiable if and only if its representative in a local coordinate patch $U, f_U(x^1, x^2, \ldots, x^n)$, is a C^∞ function on the appropriate region of E^n. But this implies consistency conditions among the various coordinate patches. On S^2 again, we might define another coordinate patch by using x and z as coordinates on the region

$$E = \{\mathbf{x} \in S^2 | y(\mathbf{x}) > 0\}$$

So the point $(x, z) \in E^2$ is made to correspond to the point

$$\mathbf{x} = (x, \sqrt{1 - x^2 - z^2}, z)$$

of $E \subset S^2$. Any function f on S^2 will have its appropriate representative

$$f_E(x, z) = f(\mathbf{x})$$

relative to the E coordinate patch. In particular, where E and N overlap, the N coordinates have E-representatives

$$x_N = x_E$$
$$y_N = \sqrt{1 - x_E^2 - z_E^2}$$

and vice-versa

$$x_E = x_N$$
$$z_E = \sqrt{1 - x_N^2 - y_N^2}$$

These equations thus define a one-to-one, differentiable transformation

$$(x_E, z_E) \to (x_N, y_N)$$

of some region of Euclidean space E^2, and its differentiable inverse. The differentiable structure of a manifold M is normally defined by covering it with coordinate patches (U, x^i). The coordinates x^i on a patch $U \subset M$ must give a one-to-one mapping $x \to (x^1(x), x^2(x), \ldots, x^n(x))$ of U onto a region of E^n. Where two patches (U, x^i), (V, y^i) overlap, i.e., on $U \cap V$, there is defined a transformation between regions of E^n,

$$(x^1(x), x^2(x), \ldots, x^n(x)) \leftrightarrow (y^1(x), y^2(x), \ldots, y^n(x))$$

This transformation must be differentiable in both directions, i.e., all the functions $x^i(y^1, \ldots, y^n)$ and $y^i(x^1, \ldots, x^n)$ must be C^∞ functions. Because the transformation is invertible, the jacobean $\partial x^i/\partial y^j$ will have a non-vanishing determinant. These consistency conditions have as an obvious consequence that if a function $f(x)$ on M is represented in one patch (U, x^i) by a C^∞ function

$$f(x) = f_U(x^1, x^2, \ldots, x^n) \text{ on } U$$

then it will be represented in (V, y^i) by a function $f_V(y^1, y^2, \ldots, y^n)$ which is C^∞ on at least that part of E^n corresponding by the $y^i(x)$ to $U \cap \dot{V}$. Thus the differentiable functions on M may be defined to be those functions $f(x)$ representable in each coordinate patch by some function $f(x^1, x^2, \ldots, x^n)$ which is differentiable on the corresponding region of E^n.

A covering of a set M by a set of consistently overlapping coordinate patches defines, as well as a differentiable structure on M, also a topology. Any subset of M which lies in a single coordinate patch and whose corresponding image in E^n is an open subset of E^n is taken to be open in M. These sets provide a basis for the open sets of M and allow us to define the continuous functions on M and other topological concepts. All the differentiable functions will be continuous.

II. Contravariant Vectors

In what follows I shall always assume that we have given some differentiable manifold M, and that all functions, mapping, curves, etc., are differentiable. Since each point $x \in M$ is contained in some coordinate patch, we can at any point introduce coordinates $x^i(x)$ for local computations. However, it is preferable to avoid using coordinates in the basic definitions so that we will not be forced to explicitly discuss the whole system of overlapping coordinate patches every time we bring up a global idea such as a vector field defined over the entire manifold.

The essential idea of a tangent vector derives from a consideration of two-dimensional surfaces imbedded in Euclidean three-space. There an arrow, attached tangent to a point of the surface, provides the same sort of coordinate free idea of a vector we have when speaking of the momentum of a particle in Newtonian mechanics. In order to avoid thinking of manifolds only as imbedded in Euclidean spaces, we want to make this arrow infinitesimal so it no longer projects out into the imbedding space. The two equivalent definitions given below are both attempts to let a tangent vector be some specific mathematical object independent of the coordinate system and related to an arrow. The first definition lets a curve in the manifold try to draw the arrow, and then makes it a small arrow by passing to infinitesimal, first derivative, properties of the curve.

Definitions: A *curve through a point* x_0 of M is a mapping $c : R \to M : t \to c(t)$ such that $c(0) = x_0$.

The *derivative* of a function f along c is the number

$$c[f] \equiv \left\{\frac{d}{dt} f(c(t))\right\}_{t=0} \tag{2.1}$$

Two curves c_1 and c_2 through x_0 *have the same tangent*, $c_1 \equiv c_2$, iff for all functions f defined each in some neighborhood of x_0 one has

$$c_1[f] = c_2[f] \tag{2.2}$$

After these preliminary definitions we now want to define a tangent vector at x_0 as the common property of all curves which proceed in the same direction and at the same rate through x_0 (as measured by functions f). The mathematical device used to abstract a common property out of many examples is an equivalence relation, such as the relation $c_1 \equiv c_2$ defined above.

Definition: A *tangent vector* $\mathbf{v}$ at x_0 is an equivalence class of curves through x_0, all having the same tangent, i.e.,

$$\mathbf{v} = \{c \mid c \equiv c_1 \quad \text{for some fixed } c_1\}$$

If $c \in \mathbf{v}$, we say that $\mathbf{v}$ is tangent to c at x_0. A tangent vector $\mathbf{v}$ can be used as a differential operator on functions by defining $\mathbf{v}[f]$, the *derivative of* f *along* $\mathbf{v}$, by the equation

$$\mathbf{v}[f] = c[f], \quad c \in \mathbf{v} \tag{2.3}$$

Because of equation (2.2), any curve with tangent $\mathbf{v}$ may be used here.

The relationship of this definition to the older, component-coordinate, definition can be seen by writing out equation (2.2) in a local coordinate system x^i where it reads

$$\left(\frac{\partial f}{\partial x^i}\right)_{x_0}\left(\frac{dc_1{}^i}{dt}\right)_0 = \left(\frac{\partial f}{\partial x^i}\right)_{x_0}\left(\frac{dc_2{}^i}{dt}\right)_0 \tag{2.4}$$

if $x^i = c_1{}^i(t)$ and $x^i = c_2{}^i(t)$ represent the two curves. Thus, all the curves with the same tangent are characterized by the same values for $(dx^i/dt)_0$ and this set of numbers, associated with the coordinate system x^i, identifies the equivalence class $\mathbf{v}$.

As a differential operator, a tangent vector $\mathbf{v}$ has the following two properties

(1) $\mathbf{v}$ is linear

$$\mathbf{v}[af + bg] = a\mathbf{v}[f] + b\mathbf{v}[g] \tag{2.5l}$$

(2) $\mathbf{v}$ is a "differentiation"

$$\mathbf{v}[fg] = f(x_0)\mathbf{v}[g] + g(x_0)\mathbf{v}[f] \tag{2.5d}$$

Here a, b are real numbers and f, g are any functions defined each in some neighborhood of x_0. The next paragraph simplifies this last remark to "$f, g \in \mathscr{F}(x_0)$".

While the domain of definition of a function is frequently important, it will also often be convenient to ignore it. This can be done for local questions in a precise way by defining:

Two functions f_1, f_2 *have the same germ at* x_0 if each is defined in some neighborhood U_1, U_2 of x_0 and if, on a neighborhood $U \subset U_1 \cap U_2$ of x_0, one has $f_1 = f_2$.

Then obtain the set $\mathscr{F}(x)$ of germs of differentiable functions at x by defining:

A *germ* f *of a differentiable function at* x is an equivalence class of functions, all having the same germ at x. I purposely use the same notation f for germs as for functions, since the context can make clear which is meant when it matters.

The differential operators $\mathbf{v}$ corresponding to tangent vectors can be combined linearly in an obvious way

$$(a\mathbf{v}_1 + b\mathbf{v}_2)[f] = a\mathbf{v}_1[f] + b\mathbf{v}_2[f] \tag{2.6}$$

For this reason it is convenient to define: A *tangent vector* $\mathbf{v}$ is a linear differentiation on $\mathscr{F}(x)$, i.e., a mapping $\mathbf{v} : \mathscr{F}(x) \to R$ satisfying equations (2.5). With this definition, which we shall find equivalent to the previous one, it is evident that the tangent vectors at x form a vector space over the real numbers.

To study the equivalence of our definitions we introduce a local coordinate system x^i. Then the curves through x_0 which follow the coordinate lines define certain tangent vectors

$$\mathbf{e}_i : f \rightarrow \left(\frac{\partial f}{\partial x^i}\right)_{x_0} \tag{2.7}$$

In terms of these differential operators $\mathbf{e}_i$ we want to compute $\mathbf{v}[f]$ for some given $\mathbf{v}$ satisfying equations (2.5). To begin we put $f(x)$ in a form which lets us use the product rule (2.5d):

$$\begin{aligned} f(x) &= f(x_0) + [f(x) - f(x_0)] \\ &= f(x_0) + \int_0^1 \frac{d}{dt} f(x_0{}^i + t(x^i - x_0{}^i))\, dt \\ &= f(x_0) + (x^i - x_0{}^i)\int_0^1 f_{,i}(x_0 + t(x - x_0))\, dt \end{aligned}$$

$$f(x) = f(x_0) + (x^i - x_0{}^i) g_i(x). \tag{2.8}$$

Here $f_{,i}$ means the partial derivative of (the coordinate representative of) f with respect to its ith argument, and

$$g_i(x) = \int_0^1 f_{,i}(x_0{}^k + t(x^k - x_0{}^k))\, dt$$

satisfies

$$g_i(x_0) = f_{,i}(x_0) = \left(\frac{\partial f}{\partial x^i}\right)_{x_0} = \mathbf{e}_i[f] \tag{2.9}$$

Acting on equation (2.8) with the operator $\mathbf{v}$ gives

$$\mathbf{v}[f] = \mathbf{v}[x_0{}^i]\mathbf{e}_i[f] \tag{2.10}$$

after using (2.5d), (2.9), and the fact that derivatives of constants vanish:

$$\mathbf{v}[a] = 0 \tag{2.11}$$

[This relation follows from equations (2.5), for with $b = 0$ in (2.5l) we have $\mathbf{v}[af] = a\mathbf{v}[f]$ while (2.5d) with $g = a$ gives

$$\mathbf{v}[af] = f(x_0)\mathbf{v}[a] + a\mathbf{v}[f] = a\mathbf{v}[f]$$

or $f(x_0)\mathbf{v}[a] = 0$. Equation (2.11) follows, since we may assume $f(x_0) = 1$.]

To show the equivalence of our two definitions of tangent vectors, we must show that any operator $\mathbf{v}$ is tangent to some curve. The equation

$$x^i(t) = x_0{}^i + t\mathbf{v}[x_0{}^i]$$

defines such a curve.

In equation (2.10) the numbers

$$v^i \equiv \mathbf{v}[x_0{}^i] \tag{2.12}$$

are the *components of* $\mathbf{v}$ with respect to the basis $\mathbf{e}_i$; because any tangent vector can be expanded as a linear combination (2.10) of the $\mathbf{e}_i$ we learn that the vector space

$T(x_0)$ of tangent vectors at x_0 is n-dimensional. Using a more specific notation

$$\mathbf{e}_i = \left(\frac{\partial}{\partial x^i}\right)_{x_0} \tag{2.13}$$

for these basis vectors we can rewrite equation (2.10) in the form

$$\mathbf{v} = v^i\left(\frac{\partial}{\partial x^i}\right)_{x_0} \tag{2.14}$$

which emphasizes the operator character of $\mathbf{v}$ while

$$\mathbf{v} = v^i\mathbf{e}_i \tag{2.15}$$

shows the usual expansion of a vector in terms of a basis. If we wish to choose some other basis, $\mathbf{b}_{k'}$, then the expansions

$$\mathbf{e}_k = A_k{}^{k'}\mathbf{b}_{k'} \tag{2.16}$$

substituted in (2.15) give

$$\mathbf{v} = v^k\mathbf{e}_k = v^k A_k{}^{k'}\mathbf{b}_{k'} = v^{k'}\mathbf{b}_{k'}$$

thus deriving the transformation law

$$v^{k'} = A_k{}^{k'}v^k \tag{2.17}$$

for the components of $\mathbf{v}$. Although a basis for $T(x_0)$ need not be obtained from a coordinate system, in case both $\mathbf{e}_k = \partial/\partial x^k$ and $\mathbf{b}_{k'} = \partial/\partial y^{k'}$ are coordinate axes we have the classical rule

$$A_k{}^{k'} = \left(\frac{\partial y^{k'}}{\partial x^k}\right)_{x_0} \tag{2.18}$$

end here →

in (2.16) and (2.17).

The preceding transformation law considered the case where we had one fixed manifold and one fixed vector $\mathbf{v}$ at x, and let the coordinate system and/or the basis for $T(x)$ shift about under our feet. The components v^i of $\mathbf{v}$ then changed although $\mathbf{v}$ itself did not. We now turn to a quite different situation, in which we wish to consider two manifolds and a mapping between them $\phi : M \rightarrow N$. Given a vector $\mathbf{v}$ at $x \in M$, we will define a corresponding vector $\phi^{\#}\mathbf{v}$ at $\phi(x) \in N$. Again there are two equivalent definitions depending on which definition of tangent vector we refer to. If c is a curve through x with tangent $\mathbf{v}$, then $\phi^*c \equiv \phi \circ c$ is a curve through $\phi(x) \equiv \phi x$; we define its tangent there to be $\phi^{\#}\mathbf{v}$ and could verify that $\phi^{\#}\mathbf{v}$ does not depend on which curve $c \in \mathbf{v}$ was used. This definition of $\phi^{\#}$ lets us visualize how vectors are transformed by imagining how $\phi : M \rightarrow N$ transformed the points of the curve which drew the arrow, $\mathbf{v}$. Somewhat more elegant is the definition of the differential operator $\phi^{\#}\mathbf{v}$ by the equation

$$(\phi^{\#}\mathbf{v})[f] = \mathbf{v}[\phi_* f] \tag{2.19}$$

or the equivalent diagram

$$\begin{tikzcd} \mathscr{F}(x) \arrow[dr, "\mathbf{v}"'] & & \mathscr{F}(\phi x) \arrow[ll, "\phi_*"'] \arrow[dl, "\phi^{\#}\mathbf{v}"] \\ & R & \end{tikzcd} \tag{2.20}$$