VAN NOSTRAND MATHEMATICAL STUDIES #3

notes on DIFFERENTIAL GEOMETRY

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1. Manifolds

In this chapter we define the fundamental concepts which we deal with throughout these notes. Specifically, the notions of manifold, function, and vector, and the concept of differentiability (smoothness), must be carefully digested for a solid foundation.

Section 1.1 *Manifolds*

First some notation. Let R be the set of real numbers. For an integer $n > 0$, let R^n be the product space of ordered n-tuples of real numbers. Thus $R^n = [(a_1,\ldots,a_n)\colon\ a_i \text{ in } R]$. For $i = 1,\ldots,n$, let u_i be the natural coordinate (slot) functions of R^n, i.e., $u_i\colon\ R^n \to R$ by $u_i(a_1,\ldots,a_n) = a_i$. An open set of R^n will be a set which is open in the standard metric topology induced by the standard metric function d on R^n; thus if $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ are points in R^n, then $d(a, b) = [\sum_{i=1}^n (a_i - b_i)^2]^{1/2}$.

The concept of differentiability is based ultimately on the definition of a derivative in elementary calculus. Let r be an integer, $r > 0$. Recall from advanced calculus that a map f from an open set $A \subset R^n$ into R is called C^r on A if it possesses continuous partial derivatives on A of all orders $\leq r$. If f is merely continuous from A to R, then f is C^0 on A. If f is C^r on A for all r, then f is C^∞ on A. If f is real analytic on A (expandable in a power series in the coordinate functions about each point of A), then f is C^ω on A. Henceforth, unless otherwise specified, *we let r be ∞, ω, or an integer* > 0.

A map f from an open set $A \subset R^n$ into R^k (k an integer ≥ 1) is C^r on A if each of its slot functions $f_i = u_i \circ f$ is C^r on A for $i = 1,\ldots,k$; thus for p in R^n, $f(p) = (f_1(p),\ldots,f_k(p))$ in R^k.

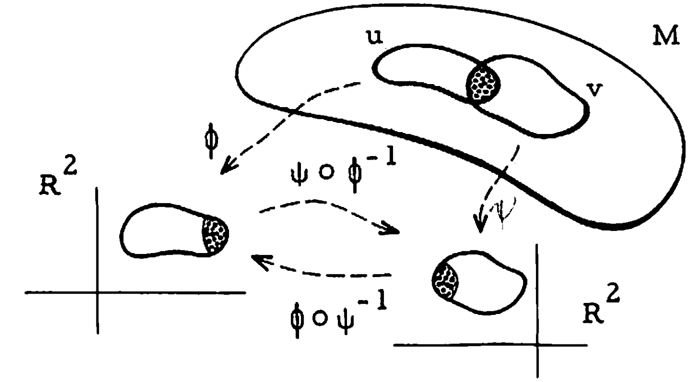

Fig. 1.1 Overlapping Coordinate Domains

We now define a manifold. Let M be a set. An *n-coordinate pair* on M is a pair (ϕ, U) consisting of a subset U of M and 1 to 1 map ϕ of U onto an open set in R^n. One n-coordinate pair (ϕ, U) is *C^r related* to another n-coordinate pair (θ, V) if the maps $\phi \circ \theta^{-1}$ and $\theta \circ \phi^{-1}$ are C^r maps wherever they are defined (thus their domains of definition must be open). A *C^r n-subatlas* on M is a collection of n-coordinate pairs (ϕ_h, U_h), each of which is C^r related to every other member of the collection, and the union of the sets U_h is M. A maximal collection of C^r related n-coordinate pairs is called a *C^r n-atlas.* If a C^r n-atlas contains a C^r n-subatlas, we say the subatl as induces or generates the atlas. Finally, an *n dimensional C^r manifold* or a *C^r n-manifold* is a set M together with a C^r n-atlas. When $r = 0$, M is customarily called a *locally Euclidean space* or a *topological manifold,* and only when $r \neq 0$ is M called a *differentiable* or *smooth* manifold. An atlas on a set M is often called a *differentiable structure* or a *manifold structure* on M. Notice that one set may possess more than one differentiable structure (see example 4 below), however, a definition of "equivalent" differentiable structures is necessary before the study of different atlases on a set becomes meaningful (see Munkres[1]).

Each n-coordinate pair (ϕ, U) on a set M induces a set of n real valued functions on U defined by $x_i = u_i \circ \phi$ for $i = 1,\dots,n$. The functions $x_1,\dots,x_n$ are called *coordinate functions* or *a coordinate system* and U is called the *domain* of the coordinate system.

We list some examples:

1. Let M be R^n with a C^r n-subatlas equal to the pair consisting of ϕ = the identity map and $U = R^n$.

2. Let M be any open set of R^n and let a $C^r n$-subatlas be (the identity map, M).

3. Let $M = GL(n, R)$, the group of non-singular R-linear transformations of R^n into itself. Then M can be mapped 1:1 onto an open set in R^{n^2} and thus a manifold structure can be defined on M via example 2. If (a_{ij}) is a matrix representation of an element of M with respect to the usual base of R^n, then map (a_{ij}) into the n^2-tuple

$$(a_{11}, a_{12},\dots,a_{1n}, a_{21}, a_{22},\dots,a_{2n}, a_{31},\dots,a_{nn}).$$

The image set of this map will be open since it is the inverse image of an open set by the determinant map, which is continuous (indeed it is C^ω as a map on R^{n^2}).

4. Let M_1 be the 1-dimensional C^1 manifold of example 1, and let $M_2 = R$ with the C^1 1-subatlas (x^3, R), where x is the identity mapping on R. Then $M_1 \neq M_2$ since $x^{1/3}$ is not C^1 at the origin.

5. Let f be a C^r real valued function on R^{n+1}, with $r > 0$ and $n > 0$, and suppose the gradient of f does not vanish on an f-constant set $M = [p \text{ in } R^{n+1}\colon\ f(p) = 0]$. Then at each point in M, choose any partial derivative of f that doesn't vanish, say the i^{th} one, apply the implicit function theorem to obtain a neighborhood of p (relative topology on M) which projects in a 1:1 way into the $u_i = 0$ hyperplane of R^{n+1}, and thus define a subatlas which makes M a C^r n-manifold.

   This example covers many classical hypersurfaces in R^{n+1}, including spheres, planes, and cylinders.

6. The process in example 5 can easily be generalized to obtain C^r $(n-k)$-manifolds from "constant sets" of a C^r map $f\colon\ R^n \to R^k$ whose Jacobian matrix is of rank k on the constant set.

7. Let F be a univalent map from an open set in R^n into R^m, with $0 < n < m$, and let M be the image of F. Then the n-coordinate pair (F^{-1}, M) defines a C^r n-subatlas on M.

For further definitions, let M be a fixed C^r n-manifold. An *open* set in M is a subset A of M such that $\phi(A \cap U)$ is open in R^n for every n-coordinate pair (ϕ, U). The reader can verify that M becomes a topological space with this definition of the open sets. If p in M, then a *neighborhood of p* is any open set containing p. Notice M need not be Hausdorff. The concept of Hausdorffness is irrelevant for much of local differential geometry. It becomes relevant in passing from a Riemannian metric to a distance function.

Section 1.2 Smooth Functions

In this section let A be the domain of a function f and assume A is an open subset of the C^r n-manifold M. If f is real valued, then f is C^s *on A* if $f \circ \phi^{-1}$ is C^s on $\phi(A \cap U)$ for every coordinate pair (ϕ, U) on M. Note the independence of r and s. If N is a C^k d-manifold and f is N-valued, and f is C^s *on A* if f is continuous and for every real valued function g, that is C^s on an open domain in N, the composite $g \circ f$ is C^s on $A \cap f^{-1}$ (domain of g). Note the independence of r, k, and s.

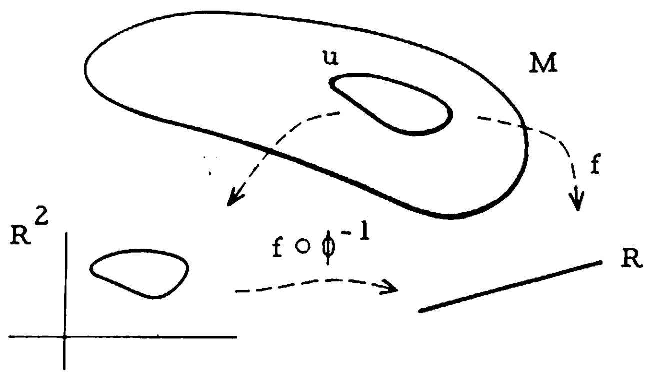

Fig. 1.2 An Induced Map from R^2 into R

The local character of the smoothness of a function is captured in the following definition. Suppose the domain of f is not necessarily open and f is N-valued. If p is in the domain of f, then f is C^s *at* p if there is a neighborhood U of p with f defined on U such that $f|_U$ is C^s on U. As a corollary, if f is C^s at every point in its domain then its domain is open.

Let us now *specialize to C^∞ manifolds and C^∞ functions.* This is done for convenience chiefly and it allows us to define a tangent vector in a very elegant way. Our concern in these notes is not with "the least possible assumptions" but rather with those concepts that arise naturally in a general situation. The restriction is not too drastic because of the following result due to Whitney: A C^r atlas on a set with $r > 0$ contains a C^∞ atlas (see Munkres[1]). There is an example of Kervaire which exhibits a C^0 atlas on a set which admits no C^1 atlas. For further work on the "equivalence" of differentiable structures see Milnor[1 and 2], Munkres[1 and 2], and Smale[1].

The following list of nine problems are recommended in order to familiarize oneself with the notion of a C^∞ map. In particular the problems are aimed at obtaining numbers 6 and 7 which are often useful. The list (remember A is open in M, which is a C^∞ n-manifold);

1. The map f: $A \to N$ is C^∞ on A iff f is C^∞ at each point p in A.
2. If f: $A \to N$, f is C^∞ on A, and U is an open set contained in A, then $f|_U$ is C^∞ on U.

3. Let U_h be a collection of open sets in M and let f_h: $U_h \to N$ be C^∞ on U_h for each h. If f is a function whose domain is the union of all U_h and if $f|_{U_n} = f_h$ for all h, then f is C^∞ on its domain.
4. If f: $A \to R^k$ is C^∞ on $A \subset R^n$ and g: $B \to R$ is C^∞ on the open set $B \subset R^k$, then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.
5. If f: $A \to N$ is C^∞ on $A \subset M$ and (ϕ, U) is a coordinate pair on M, then $f \circ \phi^{-1}$ is C^∞ on $\phi(A \cap U)$.
6. Let P be a C^∞ s-manifold. If F: $A \to N$ is C^∞ on $A \subset M$ and g: $B \to P$ is C^∞ on the open set $B \subset N$, then $g \circ f$ is C^∞ on $A \cap f^{-1}(B)$.
7. The map f: $A \to N$ is C^∞ on $A \subset M$ iff for every coordinate pair (ϕ, U) in a subatlas on N the functions $x_i \circ f$ are C^∞ on $A \cap f^{-1}(U)$, for $i = 1,\ldots,d$ and $x_i = u_i \circ \phi$.
8. If $n \geq k$ and g: $R^n \to R^k$ by $g(a_1,\ldots,a_n) = (a_1,\ldots,a_k)$ then g is C^∞ on R^n. If h: $R^k \to R^n$ by $h(a_1,\ldots,a_k) = (a_1,\ldots,a_k, 0,\ldots,0)$ then h is C^∞ on R^k.
9. Let f and g be real valued functions that are C^∞ on the subsets A and B of M, respectively. Show that $f + g$ and fg are C^∞ on $A \cap B$, where $(f + g)(p) = f(p) + g(p)$ and $(fg)(p) = f(p)g(p)$.

For the record, we can and so do define a Lie group. A *Lie group* G is a group G whose underlying set is also a C^∞ manifold such that the group operations are C^∞, i.e. the map ϕ: $G\text{x}G \to G$ where $\phi(g, h) = gh^{-1}$ is C^∞ (see problem 18 and 20).

One last bit of notation, let $C^\infty(A, N)$ denote the set of C^∞ functions mapping an open set A in a manifold M into a manifold N.

Section 1.3 Vectors and vector fields

The definition of a tangent vector generalizes the "directional derivative" in R^n. If X_m is an ordinary (advanced calculus) vector at a point m in R^n and f is a C^∞ function in a neighborhood of m, then define $X_m f = X_m \cdot (\nabla f)_m$, where ∇f is the gradient vector field of f. From the properties of the "dot" product and the operator ∇, it follows that

$$X_m(af + bg) = aX_m f + bX_m g$$

$$X_m(fg) = f(m)X_m g + g(m)X_m f,$$

where g is a C^∞ function in a neighborhood of m and a and b are real numbers. Notice X is *not* normalized to be a unit vector. We generalize now to define a tangent vector on a manifold as an operator on C^∞ functions which obeys the above rules.

Let M be a C^∞ n-manifold. Let m be in M and let $C^\infty(m)$ denote the set of real valued functions that are C^∞ on some neighborhood of m. A *tangent vector at* m is a real valued function X on $C^\infty(m)$ having the following properties:

(1) $X(f + g) = Xf + Xg$, $X(bf) = b(Xf)$

(2) $X(fg) = (Xf)g(m) + f(m)(Xg)$,

where f and g are in $C^\infty(m)$, and b is in R. The set $C^\infty(m)$ is almost a ring (there is a slight problem with domains), and thus a tangent vector is often called a derivation on $C^\infty(m)$.

The *tangent space to* M *at* m, denoted by M_m, is the set of all tangent vectors at m. It is a vector space over the real field where $(X + Y)f = Xf + Yf$ and $(bX)f = b(Xf)$ for X, Y in M_m, f in $C^\infty(m)$, and b a real number.

Let $x_1,\ldots,x_n$ be a *coordinate system about* m (i.e., m is in the domain of these coordinate functions). We define for each i, a coordinate vector at m, denoted $(\partial/\partial x_i)_m$ by

$$\left(\frac{\partial}{\partial x_i}\right)_m f = \frac{\partial(f \circ \phi^{-1})}{\partial u_i}(\phi(m))$$

where $x_i = u_i \circ \phi$ and the differentiation on the right side is as usual on R^n. The verification of properties (1) and (2) above we leave to the reader. In a moment we show these coordinate vectors form a base for the tangent space at m.

LEMMA. *Let* $x_1,\ldots,x_n$ *be a coordinate system about* m *with* $x_i(m) = 0$ *for all* i. *Then for every function* f *in* $C^\infty(m)$ *there exists* n

functions $f_1,\ldots,f_n$ *in* $C^\infty(m)$ *with* $f_i(m) = (\partial/\partial x_i)_m f$ *and* $f = f(m) + \Sigma_i x_i f_i$ *in a neighborhood of* m. (*Note the equality in question is an equality between functions, and* $f(m)$ *represents a constant function with value* $f(m)$; *the sum is taken for* $i = 1, 2,\ldots,n$, *and in the future this relevant range is to be understood.*)

Proof. Let ϕ be the coordinate map belonging to the x_i. Let $F = f \circ \phi^{-1}$, and we know F is defined in a ball about the origin in R^n, i.e., in a set $B = [p$ in R^n: distance from origin to $p < r]$. For $(a_1,\ldots,a_n)$ in B we have,

$$F(a_1,\ldots,a_n) = F(a_1,\ldots,a_n) - F(a_1,\ldots,a_{n-1}, 0) +$$

$$F(a_1,\ldots,a_{n-1}, 0) - F(a_1,\ldots,a_{n-2}, 0, 0) +\ldots+$$

$$F(a_1, 0,\ldots,0) - F(0,\ldots,0) + F(0,\ldots,0) =$$

$$= \Sigma_i F(a_1,\ldots,a_{i-1}, ta_i, 0,\ldots,0)]_0^1 + F(0,\ldots0)$$

$$= F(0,\ldots,0) + \Sigma_i \int_0^1 \frac{\partial F}{\partial u_i}(a_1,\ldots,a_{n-1}, ta_i, 0,\ldots,0)a_i dt$$

$$= F(0,\ldots,0) + \Sigma_i a_i F_i(a_1,\ldots,a_n), \text{ where}$$

$$F_i(a_1,\ldots,a_n) = \int_0^1 \frac{\partial F}{\partial u_i}(a_1,\ldots,a_{i-1}, ta_i, 0,\ldots,0)dt$$

is C^∞ in B since $(\partial F/\partial u_i)$ is C^∞. Let $f_i = F_i \circ \phi$ and the lemma is proved. //

THEOREM. *Let* M *be a* C^∞ n*-manifold and let* $x_1,\ldots,x_n$ *be a coordinate system about* m *in* M. *Then if* X *in* M_m, $X = \Sigma_i(Xx_i)(\partial/\partial x_i)_m$, *and the coordinate vectors form a base for* M_m *which thus has dimension* n.

Proof. We first prove the stated representation. Take X in M_m and f in $C^\infty(m)$. If $x_i(m) \neq 0$ for all i, let $y_i = x_i - x_i(m)$. Then apply the lemma to f with respect to the coordinate system $y_1,\ldots,y_n$ and notice $(\partial f/\partial y_i)(m) = (\partial f/\partial x_i)(m)$. Next we see if c a constant map then

$$X(c) = cX(1) = c(1X(1) + 1X(1)) = 2cX(1)$$

which implies $cX(1) = 0$ and $X(c) = 0$. Thus $Xf = X(f(m) + \Sigma_i y_i f_i)$

$$= \Sigma_i[(Xy_i)f_i(m) + y_i(m)(Xf_i)]$$

$$= \Sigma_i X(x_i - x_i(m))f_i(m)$$

$$= \Sigma_i (Xx_i)(\partial f/\partial x_i)(m)$$

which proves the required representation. If $Y = \Sigma_i a_i(\partial/\partial x_i) = 0$ then $0 = Yx_j = a_j$, thus the coordinate vectors are independent and span M_m.//

A *vector field* X on a set A is a mapping that assigns to each point p in A a vector X_p in M_p. A field X is C^∞ on A if A is open and for each real valued function f that is C^∞ on B, the function $(Xf)(p) = X_p f$ is C^∞ on $A \cap B$. If X and Y are C^∞ vector fields on A their *bracket* is a C^∞ vector field $[X, Y]$ on A defined by $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$.

If f and g are C^∞ functions, it is trivial that $[X, Y](f + g) = [X, Y]f + [X, Y]g$, and $[X, Y](af) = a[X, Y]f$ for a in R. To check the product property, consider

$$[X, Y](fg) = X(Y(fg)) - Y(X(fg))$$

$$= X(fYg + gYf) - Y(fXg + gXf)$$

$$= fXYg + (Xf)(Yg) + (Xg)(Yf) + gXYf$$

$$- fYXg - (Yf)(Xg) - (Yg)(Xf) - gYXf$$

$$= f[X, Y]g + g[X, Y]f.$$

Thus $[X, Y]$ is a vector field and the proof of its C^∞ nature we leave as a problem.

For later use, notice that $[X, Y] = -[Y, X]$, $[X, X] = 0$, and the bracket is linear in each slot with respect to addition, i.e., $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$. However, $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$, and it is this property that prevents the bracket mapping from being a tensor (problem 10). Problem 13 gives a geometric interpretation of the bracket, and in section 9.1 there are applications involving integrability conditions. For example, if $x_1,\ldots,x_n$ is a coordinate system then $[\partial/\partial x_i, \partial/\partial x_j] = 0$ for all i and j (since cross partial derivatives of C^∞ functions are equal), and actually this condition on n independent vector fields is sufficient to imply the fields are coordinate vector fields (section 9.1).

The bracket operation also satisfies the following expression which is called the *Jacobi identity*,

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

where X, Y, and Z are C^∞ fields with a common domain.

Section 1.4 The Jacobian of a map

Let M and N be C^∞ manifolds of dimensions n and k respectively. We defined above the concept of a C^∞ map f from M into N. Such a map induces a linear transformation from each tangent space M_m into the tangent space $N_{f(m)}$. This linear map is called the *Jacobian map* or the *differential of* f and we denote it by f_* (often it is denoted df, but we reserve the symbol d for the exterior derivative operator). Let X be in M_m and we define f_*X as a vector at $f(m)$ in N by taking a function g which is C^∞ in a neighborhood of $f(m)$ and setting $(f_*X)g = X(g \circ f)$. It is trivial to check that f_*X is a vector at $f(m)$ and the map f_* is linear.

By selecting a coordinate system $x_1,\ldots,x_n$ about m and another $y_1,\ldots,y_k$ about $f(m)$, we can determine a matrix representation for f_* which is called the *Jacobian matrix* of f_* with respect to the chosen coordinate systems. Let $X_i = \partial/\partial x_i$, $Y_j = \partial/\partial y_j$, thus $X_1,\ldots,X_n$, at m, form a base for M_m and we compute f_* by computing its action on this base. Namely, $f_*X_i = \Sigma_j(f_*X_i)y_jY_j$ by the representation theorem above, hence the matrix in question is the matrix $((f_*X_i)y_j) = (\partial(y_j \circ f)/\partial x_i)$ for $1 \leq i \leq n$ and $1 \leq j \leq k$.

The implicit function theorem and the inverse function theorem can be applied and formulated in this language. The former we postpone, since we do not really need it for some time (see problem 16) but the latter is both useful and instructive. First a definition. A *diffeomorphism* is a map f: $M \to N$ that is 1:1 and onto with both f and f^{-1} C^∞, and if such an f exists, then M is *diffeomorphic* to N.

THEOREM. (*Inverse function*) *Let M and N be C^∞ n-manifolds and let f: $M \to N$ be C^∞. If for m in M, the Jacobian f_* at m is an isomorphism of M_m onto $N_{f(m)}$ then there is a neighborhood U of m and a neighborhood V of $f(m)$ such that f is a diffeomorphism from U to V (i.e., f is a local diffeomorphism about m).*

We leave it to the reader to choose a coordinate system on both sides and apply the theorem from advanced calculus to obtain the result. Notice the C^∞ demand of f *and* f^{-1} implies the theorem could be stated as a *necessary* as well as a sufficient condition for the existence of a local inverse. If one only demands continuity of the inverse, then the map $x \to x^3$ provides a homeomorphism of R onto R whose Jacobian is singular at the origin.

Now consider the behavior of the Jacobian with respect to composite maps. Let g be a C^∞ map of N into the C^∞ manifold L. Then at each m in M, $(g \circ f)_* = g_* \circ f_*$, for if h is a C^∞ function about $g(f(m))$ and X in M_m then $((g \circ f)_*X)h = X(h \circ g \circ f) = (f_*X)(h \circ g) = (g_*(f_*X))h$. In terms of coordinate systems, the above computation exhibits the chain rule and a multiplicative behavior of Jacobian matrices. When f is a diffeomorphism of M into N, and X and Y are C^∞ fields on M, then f_*X and f_*Y are C^∞ fields on N with $f_*[X, Y] = [f_*X, f_*Y]$.

Section 1.5 *Curves and integral curves*

In these notes curves will be viewed as a special case of mappings, thus we will deal with "parameterized curves" almost exclusively. A *curve* in M is a C^∞ map σ from an open subset of R into M. Often we speak of a curve σ from $[a, b]$ into M where $[a, b]$ is a closed interval of real numbers, and in this case it is assumed the domain of σ is actually an open set in R containing $[a, b]$.

Let σ be a curve in M with domain U. For each t in U define the *tangent of σ at t* to be the vector $T(t)$, or $T_\sigma(t)$, at $\sigma(t)$ where $T(t) = \sigma_*(d/dt)_t$ and d/dt denotes the usual differentiation operator of real valued C^∞ functions on R. Thus if $x_1,\dots,x_n$ a coordinate system about $\sigma(t)$, then $T(t) = \Sigma_i(d(x_i \circ \sigma)/dt)_t(\partial/\partial x_i)_{\sigma(t)}$. By differentiating the coordinate parameter functions $x_i \circ \sigma(t)$ one determines the coefficients of $T(t)$ with respect to the coordinate vectors associated with the co-

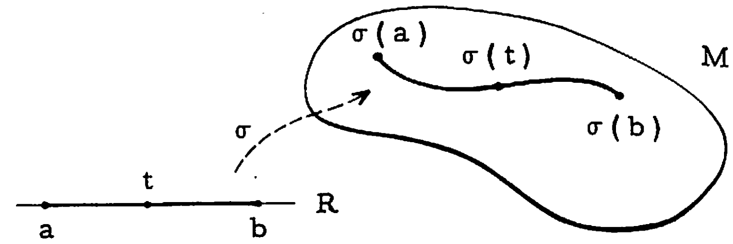

Fig. 1.3 A Curve

ordinate system. Notice this $T(t)$ is the usual "velocity" vector associated with a parameterized curve in R^3.

Having the idea of curve and tangent vector we can give a geometric description of the Jacobian f_* associated with the map f: $M \to N$. For X in M_m choose any curve σ on M with $\sigma(0) = m$ and $T_\sigma(0) = X$. Then $f \circ \sigma$ is a curve on N with $f \circ \sigma(0) = f(m)$ and indeed $f_*X = T_{f\circ\sigma}(0)$. Thus we "fill in the vector by a curve, map the curve to N, and take the new tangent vector." This device is very useful if one knows geometrically the behavior of certain curves; e.g., let $M = [(x, y, z) \text{ in } R^3\text{: } x^2 + y^2 = 1]$, let S be the unit sphere in R^3, and let f: $M \to S$ by $f(x, y, z) = (x, y, 0)$. The particular f just defined is called the "sphere map" or the "Gauss map" from M to S, since it essentially uses a unit normal vector field to M in its definition. Its Jacobian should be trivial to compute at each point from the above remarks.

We carry the idea of "filling in a vector" to a classical setting. Let X be a C^∞ vector field on the manifold M. A curve σ is an *integral curve* of X if whenever $\sigma(t)$ is in the domain of X then $T_\sigma(t) = X_{\sigma(t)}$. Thus we say the curve σ "fits" X, and suggest the physical example of the velocity vector field (which gives X) of a steady fluid flow and its streamlines (which give integral curves). The local existence of integral curves is guaranteed by the theory of ordinary differential equations.

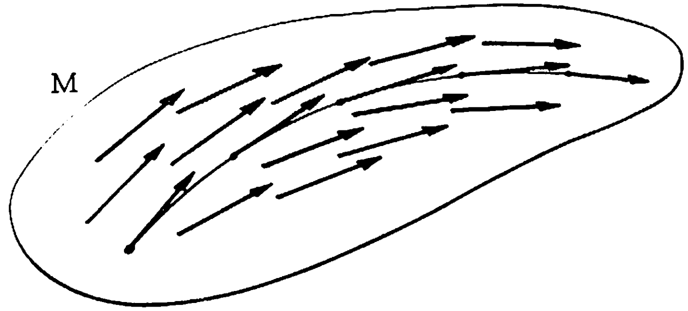

Fig. 1.4 An Integral Curve of a Vector Field

THEOREM. *Let X be a C^∞ vector field on M and let m be a point in the domain of X. Then for any real number b there exists a real number $r > 0$ and a unique curve σ: $(b - r, b + r) \to M$ such that $\sigma(b) = m$ and σ an integral curve of X.*

Proof. Let $x_1,\ldots,x_n$ be a coordinate system about m whose domain U is contained in the domain of X. Let $X = \Sigma_i f_i(\partial/\partial x_i)$ define C^∞ real valued functions f_i on U. Then the condition that a curve σ be an integral curve of X becomes the condition

$$\frac{d(x_i \circ \sigma)}{dt} = f_i \circ \sigma$$

on the domain of σ, or writing (improperly) as usual $x_i(t) = x_i \circ \sigma(t)$, we have the system of first order ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x_1,\ldots,x_n),$$

for $i = 1,\ldots,n$. Apply an existence and uniqueness theorem from differential equation theory to obtain $r > 0$ and functions $x_i(t)$ that define σ on the specified range with the required properties.//

Actually the theorem from differential equations gives much more than the above conclusion for it includes the C^∞ dependence of solutions as we vary the initial parameter b and the point m (see section 9.3). We return to this later when discussing the existence of geodesics and the exponential map (sections 5.1 and 9.3). For global ramifications see Palais[2] or Lang.

It is convenient to define a *broken C^∞ curve σ on an interval* $[a, b]$ to be a continuous map σ from $[a, b]$ into M which is C^∞ on each of a finite number of subintervals $[a, b_1]$, $[b_1, b_2],\ldots,[b_{k-1}, b]$.

Section 1.6 Submanifolds

A C^∞ k-manifold M is a *submanifold* of a C^∞ n-manifold $\overline{M}$ if for every point p in M there is a coordinate neighborhood $\overline{U}$ of $\overline{M}$ with coordinate functions $\overline{x}_1,\ldots,\overline{x}_n$ such that the set $U = [m$ in $\overline{U}$: $\overline{x}_{k+1}(m) = \ldots = \overline{x}_n(m) = 0]$ is a coordinate neighborhood of p in M with coordinate functions $x_1 = \overline{x}_1|_U,\ldots,x_k = \overline{x}_k|_U$. These coordinate systems are called *special* or *adapted* coordinate systems.

Notice it is not required that $M \cap \overline{U} = U$ so "slices" of M may approach other "slices" of M in $\overline{M}$ (see problem 17), and hence the topology on M may not be the relative topology. The definition of submanifold implies M is a subset of $\overline{M}$ and $k \leq n$. Letting i: $M \to \overline{M}$ be the inclusion map, then i is C^∞ since $\overline{x}_j \circ i$ are C^∞ maps for all special coordinate functions. The inclusion map is also an imbedding (see below) since the Jacobian i_* is non-singular, i.e., $i_*(\partial/\partial x_j(p) = \partial/\partial\overline{x}_j(p)$ for $j = 1,\ldots,k$. In these notes we will identify a tangent vector X in M_p with its image in $\overline{M}_p$ unless there is a possibility of confusion (just as we identify p and $i(p)$).

To make some more standard definitions, let M and $\overline{M}$ be C^∞ manifolds and let f be a C^∞ map of M into $\overline{M}$. If f_* is non-singular (thus f_* has no kernel) at each point p of M, then f is called an *immersion* of M into $\overline{M}$. If in addition, f is univalent, then f is called an *imbedding* of M into $\overline{M}$. A subset M' of $\overline{M}$ is called an *immersed submanifold* if there exists a manifold M and an immersion f : $M \to \overline{M}$ such that $f(M) = M'$. (Thus an immersion is a "local imbedding with self-intersections.") One can verify (problem 17) that if f: $M \to \overline{M}$ is an imbedding and $M' = f(M)$, then by defining a differentiable structure on M' so f becomes a diffeomorphism, M' becomes a submanifold of $\overline{M}$ (see Helgason, p. 23).

For examples of submanifolds see the examples 5, 6, and 7 at the end of section 1.1.

It is convenient to define a *base field* on a set A contained in an n-manifold to be a set of n vector fields that are independent at each point of A. When each field in a base field is C^∞, then the base field is C^∞. Since a set of coordinate fields is a C^∞ base field on the coordinate domain, we know C^∞ base fields always exist locally. A C^∞ base field does *not* necessarily exist over a whole manifold (consider the 2-sphere, S^2); indeed, the manifold is called *parallelizable* if it admits a global C^∞ base field.

We now define a concept which we will often use. Let M be a submanifold of $\bar{M}$ as described above. An *$\bar{M}$-vector field* Z that is C^∞ *on* M (or C^∞ on an open set A in M) is a map that assigns to each p in M (or p in A) a vector Z_p in $\bar{M}_p$ such that if $X_1,\ldots,X_n$ is any C^∞ base field on a neighborhood $\bar{U}$ of p and $Z_m = \Sigma_i^n a_i(m)(X_i)_m$ for m in $M \cap \bar{U}$ then the real valued functions a_i are C^∞ on $M \cap U$ for all i. Notice Z_p is *not* necessarily tangent to M. Since the restriction to M, of a C^∞ function on $\bar{M}$, is a C^∞ function on M, it follows if Z is C^∞ on $\bar{M}$ then $Z|_M$ is an $\bar{M}$-vector field that is C^∞ on M.

Problems (For problems 1 thru 9 see pages 4 and 5)

10. Let $W_1,\ldots,W_n$ be a C^∞ base field on an open set U in a manifold M and let $X = \Sigma_{i=1}^n f_i W_i$ be a vector field on U. Show X is C^∞ on U iff the functions f_i are C^∞ on U for all i. If Y and Z are C^∞ fields on U show $[Y, Z]$ is C^∞. Show that a coordinate field $\partial/\partial x_i$ is C^∞ on its domain. If X_p is a given vector at p in M show there is a C^∞ field $\bar{X}$ on a neighborhood of p with $\bar{X}_p = X_p$. If $x_1,\ldots,x_n$ is a coordinate system with domain U and $A = \Sigma a_i(\partial/\partial x_i)$ and $B = \Sigma b_j(\partial/\partial x_j)$ are C^∞ fields on U then find the representation of $[A, B]$ in terms of the coordinate vector fields. Show $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$ where X and Y are C^∞ fields on U and f and g are in $C^\infty(U, R)$. Prove the Jacoby identity.

11. Let A, B and C be in $C^\infty(R^3, R)$ with $B \neq 0$ anywhere. Let $V = Ai + Bj + Ck$, $X = -Bi + Aj$, and $Y = -Cj + Bk$ (advanced calculus notation). For p in R^3, let $P_p = [Z \text{ in } (R^3)_p\colon Z \cdot V_p = 0]$. Show P_p is a two-dimensional space of vectors at each point by showing X_p and Y_p are a base for P_p. Show $[X, Y]_p$ lies in P_p iff $V_p \cdot (\text{curl } V)_p = 0$. If there is a function f in $C^\infty(R^3, R)$ with grad $f \neq 0$ such that P_p is the tangent plane to the constant surface of f thru p show $V_p \cdot (\text{curl } V)_p = 0$ (see section 9.1).

Instead of seeking surfaces that are orthogonal to V (as above), one could seek surfaces whose tangent plane contains V and then one has a "geometric quasi-linear partial differential equation of the first order." Integral curves of V are called *characteristics* of the "equation." One generates solution surfaces by taking a non-characteristic curve (an "initial value" curve) and considering the surface formed by characteristics thru the initial value curve. Show two solution surfaces must intersect along a characteristic. Show there are an infinite number of solution surfaces thru one characteristic. Can there there be an initial value curve with no solution thru it?

12. Let $f\colon R^2 \to R^2$ by $f(a, b) = (a^2 - 2b, 4a^3b^2)$ and let $g\colon R^2 \to R^3$ by $g(u, v) = (u^2v + v^2, u - 2v^3, ve^u)$. Compute a matrix for f_* at $(1, 2)$ and g_* at any (u, v). Find $g_*(4\partial/\partial x - \partial/\partial y)_{(0,1)}$. Find integral curves for the vector field $X = yi + yj + 2k$ on R^3. Find a coordinate system x_1, x_2, x_3 on R^3 such that $\partial/\partial x_1 = 2i + 3j - k$ at all points.

13. Let X and Y be C^∞ fields about m in M. For small $t \geq 0$ define the curve $\sigma(t)$ as follows: go t parameter units on X integral curve thru m to p_1, go t units on Y curve thru p_1 to p_2, go t units on $(-X)$ curve thru p_2 to p_3, go t units on $(-Y)$ curve thru p_3 to $\sigma(t)$. If $\gamma(t) = \sigma(\sqrt{t})$ show $T_\gamma(0) = [X, Y]_m$. (Hint: use the lemma in section 9.1 and partial Taylor series.)

14. Let M and N be manifolds with M connected and let f and g be C^∞ maps of M into N. Show $f_* \equiv 0$ iff f is a constant map. If $f(m) = g(m)$ at one m in M and $f_* \equiv g_*$ at all points show $f = g$.

15. Let f be in $C^\infty(M, R)$ and define the *differential of f*, *df*, to be the linear map of M_m into R where $(df)_m(X_m) = X_m f$. Show $f_*(X_m) = [(df)_m(X)](\partial/\partial t)$ where t is the identity coordinate function on R. It is because of this case that in a general case the Jacobian f_* is often called the "differential of f".

16. Prove the Inverse Function Theorem (p. 10). State and prove a version of the Implicit Function Theorem of advanced calculus in terms of the Jacobian map.

17. Prove the last sentence in the third paragraph of section 1.6. Show that the image of a regular ($\sigma_* \neq 0$) univalent curve σ mapping an open interval into a manifold M is a one-dimensional submanifold of M. Let X be a unit constant vector field on R^2 with irrational slope. Let T be the set of equivalence classes on R^2 where $(a, b) \sim (c, d)$ iff $a - c = n$ and $(b - d) = m$ for integers m and n. Show T is a two-dimensional manifold (which is called the *flat torus*) in a natural way. Show X induces a vector field on T such that the image of one integral curve of X defines a one-dimensional submanifold of T that is dense in T.

18. Let M_1 and M_2 be C^∞ manifolds. Let $\pi_i\colon M_1 \times M_2 \to M_i$ by $\pi_i(m_1, m_2) = m_i$ for $i = 1, 2$. Define a C^∞ structure on $M_1 \times M_2$ so π_i are C^∞. Show $(M_1 \times M_2)_{(m_1, m_2)}$ is naturally isomorphic to $(M_1)_{m_1} \times (M_2)_{m_2}$.

19. Let M be a C^∞ n-manifold. Let $T(M) = [(m, X)\colon X \text{ in } M_m]$, and let $\pi\colon T(M) \to M$ by $\pi(m, X) = m$. If (ϕ, U) is a coordinate pair on M with $x_i = u_i \circ \phi$ let $\bar{U} = \pi^{-1}(U)$, $\bar{x}_i = x_i \circ \pi$, and for (m, X) in $\bar{U}$ let $\dot{x}_i(m, X) = a_i$ if $X = \Sigma a_i(\partial/\partial x_i)$. Let $\bar{\phi}\colon \bar{U} \to R^{2n}$ so $u_i \circ \bar{\phi} = \bar{x}_i$ and $u_{n+i} \circ \bar{\phi} = \dot{x}_i$ for $i = 1,\ldots,n$. Show the subatlas of pairs $(\bar{\phi}, \bar{U})$ defines a C^∞ structure on $T(M)$ which is called the *tangent bundle of M*. If f is a C^∞ map of M into N show f_* induces a C^∞ map of $T(M)$ into $T(N)$.

20. Let G be a Lie group. If g in G let L_g, R_g, and A_g denote the maps of G into G defined by $L_g(h) = gh$, $R_g(h) = hg$, and $A_g(h) = ghg^{-1}$. Show L_g, R_g, and A_g are C^∞. A vector field X on G is *left invariant* if $(L_g)_* X_g = X_{gh}$ for all g and h. Show a left invariant field is C^∞ and is completely determined by its value at the identity e. If X and Y are left invariant, show $[X, Y]$ is left invariant. The set of left invariant vector fields on G forms an n dimensional vector space called the *Lie algebra of* G which is denoted by $\mathfrak{g}$. Define a *one-parameter* subgroup of G to be the image of a C^∞ homormorphism of R into G. Show there is a 1:1 correspondence between one-parameter subgroups and integral curves of left invariant vector fields thru e. Show the map $(g, h) \to gh^{-1}$ is C^∞ from $G \times G$ into G iff the maps $(g, h) \to gh$ and $g \to g^{-1}$ are C^∞.

21. Let $G = GL(n, R)$ and for a matrix g in G let $u_{ij}(g) = g_{ij}$ (see example 3). Call u_{ij} the natural coordinate functions on G. Write $u_{ij} \cdot L_g$ as a linear combination of the natural coordinate functions. Let X_{ij} be the unique left invariant field on G with $X_{ij}(e) = (\partial/\partial u_{ij})(e)$ where e is the identity element. Compute X_{ij} as a field on G in terms of the coordinate vector fields. Compute $[X_{ij}, X_{rs}]$. If $A(t)$ is a C^∞ curve in G with $A(0) = e$ and $A(t)$ orthogonal for all t show $dA/dt = (da_{ij}/dt)$ is a skew-symmetric matrix for $t = 0$.

22. Let M be a C^∞ n-manifold. Let $B(M) = [(m; e_1,\ldots,e_n)\colon m \text{ in } M$ and $e_1,\ldots,e_n$ an ordered base of $M_m]$. Let $\pi\colon B(M) \to M$ by $\pi(m; e_1,\ldots,e_n) = m$. If (ϕ, U) a coordinate pair on M with $x_i = u_i \cdot \phi$, let $(\bar{\phi}, \bar{U})$ be a coordinate pair on $B(M)$ with $\bar{U} = \pi^{-1}(U)$ and $\bar{\phi}\colon \bar{U} \to R^{n+n^2}$ by the coordinate functions $\bar{x}_1,\ldots,\bar{x}_n$, $x_{11}, x_{12},\ldots,x_{nn}$ where $\bar{x}_i = x_i \cdot \pi$ and if $b = (m; e_1,\ldots,e_n)$ then $e_j = \Sigma_{i=1}^n x_{ij}(b)(\partial/\partial x_i)$. Show the subatlas of pairs $(\bar{\phi}, \bar{U})$ defines a C^∞ structure on $B(M)$ which is called the *bundle of bases over M*. For g in $GL(n, R)$ let $R_g\colon B(M) \to B(M)$ by $R_g(b) \equiv bg \equiv (m; \Sigma_{i=1}^n g_{i1}e_i, \Sigma_i g_{i2}e_i,\ldots,\Sigma_i g_{in}e_i)$ if $b = (m; e_1,\ldots e_n)$. Show R_g is C^∞. Let $s_U\colon U \to B(M)$ by $s_U(m) = (m; (\partial/\partial x_1)_m, \ldots, (\partial/\partial x_n)_m)$ for m in U. Show s_U is C^∞ and $\pi \cdot s_U$ is the identity on U. The map s_U is called the *coordinate section map over U*. Let $\hat{\phi}\colon U \times GL(n, R) \to \bar{U}$ by $\hat{\phi}(m, g) = R_g \cdot s_U(m) = s_U(m)g$. Show $\hat{\phi}$ is a diffeo onto its image. The map $\hat{\phi}$ is called a *strip map*. If (ϕ, U) and (ψ, V) are coordinate pairs on M define $s_{UV}\colon U \cap V \to GL(n, R)$ by $s_{UV}(m) = g$ if $s_U(m)g = s_V(m)$. Show s_{UV} is C^∞; it is called a *structural function* for $B(M)$. Show $(bg_1)g_2 = b(g_1 g_2)$ which justifies the name *right action* for R_g. For fixed b in $B(M)$ let $f_b\colon GL(n, R) \to B(M)$ by $f_b(g) = bg$. Show f_b is C^∞. Call the set $F_m = \pi^{-1}(m)$ the (vertical) *fiber over m* in M. Show F_m is an n^2 – submanifold of $B(M)$ and f_b is a diffo of $GL(n, R)$ onto $F_{\pi(b)}$. If $\pi(b) = \pi(c)$, show $f_c^{-1} \circ f_b$ is a left translation on $GL(n, R)$. A vector X on $B(M)$ such that $\pi_*(X) = 0$ is called a *vertical* vector. For b in $B(M)$, let $E_{ij}(b) = (f_b)_* X_{ij}(e)$ define a vector $E_{ij}(b)$ (see problem 21). Show E_{ij} is a global C^∞ vertical vector field on $B(M)$. Compute $[E_{ij}, E_{rs}]$.