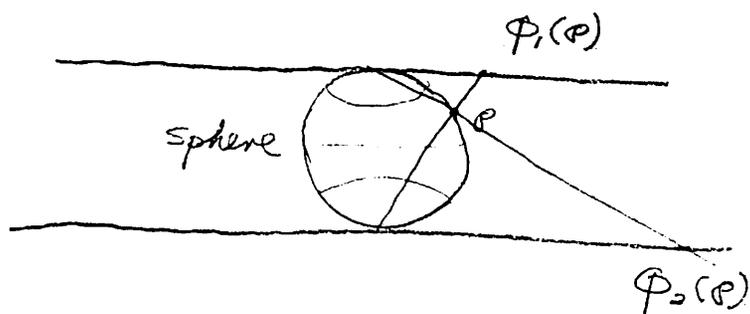


Lecture 21

1. Scalar functions with the manifold as their domain.
2. Coordinate representative of a function.

21.1

Example of a manifold with two coordinate charts



21.2 8

Example 2 (Real $n \times n$ non-singular matrices)

Let $M = \{A\} \equiv \{[a_{ij}]\}$ = set of all $n \times n$ real

Let $\Delta: \mathbb{R}^n \rightarrow \mathbb{R}$ be the determinant fn; $\Delta = \det[a_{ij}]$ non-singular matrices.

Let $A^{-1}(\mathbb{R}^n \setminus \{0\})$ = the set of all matrices having non-zero determinant.
 M is a manifold.

(i) The chart (ϕ, U) with

$\phi: U = M \rightarrow \phi(U)$ open in \mathbb{R}^{n^2}

$A \mapsto \phi(A) = (\phi_{11}(A) = a_{11}, \phi_{12}(A) = a_{12}, \dots, \phi_{nn}(A) = a_{nn})$

(Remark: $\phi(U) = \mathbb{R}^{n^2}$)

is a globally defined coordinate chart ("system")

(ii) The chart $(\bar{\phi}, \bar{U})$ with

$\bar{\phi}: \bar{U} = M \rightarrow \bar{\phi}(U)$ open in \mathbb{R}^{n^2}

$A \mapsto \bar{\phi}(A) = (\bar{\phi}_{11}(A) = a_{12}, \bar{\phi}_{12}(A) = a_{11}, \dots)$

(Remark: $\bar{\phi}(U) = \mathbb{R}^{n^2}$)

is another coordinate system

(iii) These two charts are C^∞ related because the transition map $\bar{\phi} \circ \phi^{-1}: \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n^2}$ what is this??
A: it is A.

$$(a_{11}, a_{12}, \dots) \mapsto \bar{\phi} \circ \phi^{-1}(a_{11}, a_{12}, \dots) = (a_{12}, a_{11}, \dots)$$

is infinitely differentiable. Indeed its Jacobian

$$J_j^i(\bar{\phi} \circ \phi^{-1}) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} =$$

2.1.3 4

A key attribute of a manifold, say M , is that it serves as the domain of real valued functions. Let us develop their definition.

Definition (C^s function)

a) Let M be a manifold. A real valued function

$$M \xrightarrow{f} \mathbb{R}^1$$

is said to be a C^s -function ($s \leq r$), denoted by $f \in C^s(M, \mathbb{R}^1)$, if for each chart (ϕ, U) the "coordinate representative"

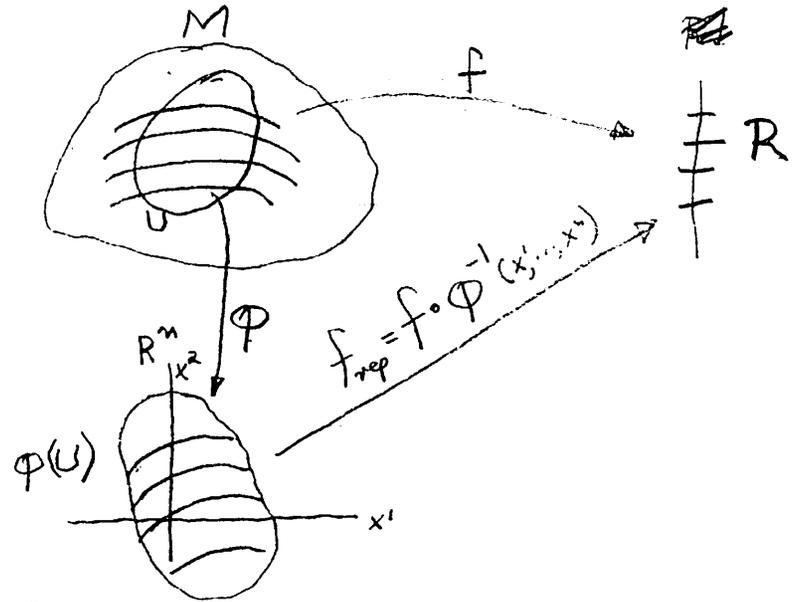
$$f_{\text{rep}}(x^1, \dots, x^m) \equiv f \circ \phi^{-1}(x^1, \dots, x^m)$$

has partial derivatives which are continuous up to order s .

Please note that this definition is the non-linear analogue of the coordinate representative introduced on Page 20.12 in the context of linear mathematics.

$f \in C^s(M, \mathbb{R}^1)$

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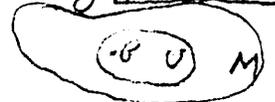
(The coordinate representative (relative to ϕ)
 $f \circ \phi^{-1}(x^1, \dots, x^m) \equiv f_{\text{rep}}(x^1, \dots, x^m)$ (for nbhd U)
of a function f .)

b) A real valued function f is said to be

" C^s in a neighborhood of P ", denoted by $f \in C^s(M, P, \mathbb{R}^1)$,

if

$$U = (\text{domain of } \phi) \subset M$$



is an open set containing P , and $f \in C^s(U, \mathbb{R}^1)$

21.5-5

Comments:

We are able to define C^s functions on M because (1) M looks locally like \mathbb{R}^n and we know about C^s function on \mathbb{R}^n .

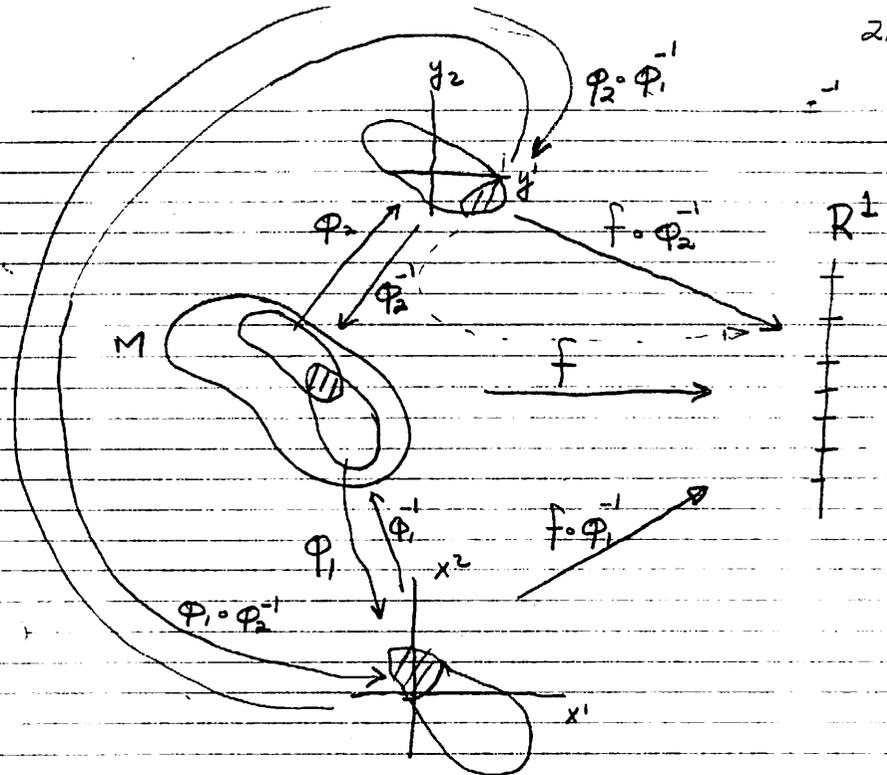
(2) if

$$U_1 = \text{domain } \varphi_1$$

$$U_2 = \text{domain } \varphi_2$$

$$U_1 \cap U_2 \neq \emptyset$$

then the concept of C^s function on $U_1 \cap U_2$ is the same relative to φ_1 as it is relative to φ_2 because $\varphi_1 \circ \varphi_2^{-1}$ is a C^r homeomorphism.



21.6

Figure 2: The essentials of the conceptual

framework surrounding an n -dimensional manifold are the same as those surrounding an n -dimensional vector space on

Page 20.14

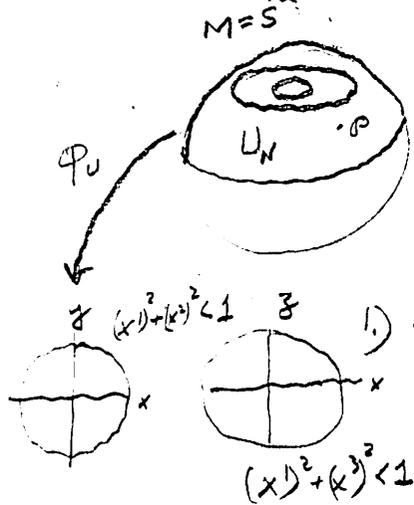
$$f \in C^s(M, \mathbb{R}^1): f \circ \varphi_2^{-1}(y^1, \dots, y^n) \equiv f_{U_2}(y^1, \dots, y^n)$$

$$(y^1, \dots, y^n) = \varphi_2 \circ \varphi_1^{-1}(x^1, \dots, x^n) \text{ is } C^r \text{ because } M \text{ is a } C^r \text{ manifold}$$

$$f \in C^s(M, \mathbb{R}^1): f \circ \varphi_2^{-1} \circ \varphi_2 \circ \varphi_1^{-1}(x^1, \dots, x^n) = f \circ \varphi_1^{-1}(x^1, \dots, x^n) \equiv f_{U_1}(x^1, \dots, x^n)$$

because $s \leq r$

Example 3 (Unit 2-sphere S^2) 21.7 8



$$p = (x, y, z)$$

Define three real valued functions and let $M = S^2$

$$\begin{aligned} x^1 &= x(p) \\ x^2 &= y(p) \\ x^3 &= z(p) \end{aligned} \quad S^2 = \{p \mid x^2(p) + y^2(p) + z^2(p) = 1\}$$

1) Northern Hemisphere:

$$U_N = \{p \in S^2 \mid z(p) > 0\}$$

$$\varphi_N: U_N \rightarrow \mathbb{R}^2$$

$$p = (x, y, \sqrt{1-x^2-y^2}) \xrightarrow{\varphi_N} \varphi_N(p) = (x, y)$$

For any f defined on U_N we have

$$f(p) = f(x, y, z) = f(x, y, \sqrt{1-x^2-y^2}) = f_{N, \text{rep}}(x, y)$$

$$= f \circ \varphi_N^{-1}(x, y)$$

2) Eastern Hemisphere

$$\text{Let } U_E = \{p \in S^2 \mid y(p) > 0\}$$

$$\varphi_E: U_E \rightarrow \mathbb{R}^2$$

$$p = (x, \sqrt{1-z^2}, z) \xrightarrow{\varphi_E} \varphi_E(p) = (x, z)$$

For $p \in U_E$

$$f(p) = f(x, z, z) = f(x, \sqrt{1-z^2}, z) = f_{E, \text{rep}}(x, z) = f \circ \varphi_E^{-1}(x, z)$$

3) Transition map:

$$\varphi_N \circ \varphi_E^{-1}: (x, z) \mapsto (x, y) \quad \begin{aligned} x_N &= x_E \\ y_N &= \sqrt{1-x_E^2-z_E^2} \end{aligned}$$

21.8 9

$$\varphi_E \circ \varphi_N^{-1}(x, y) : \begin{aligned} x_E &= x_N \\ z_E &= \sqrt{1-x_N^2-y_N^2} \end{aligned}$$

$$\varphi_N: (x, y, \sqrt{1-x^2-y^2}) \mapsto \varphi_N(x, y, \sqrt{1-x^2-y^2}) = (x, y)$$

$$\varphi_N^{-1}: (x, y) \mapsto \varphi_N^{-1}(x, y) = (x, y, \sqrt{1-x^2-y^2})$$

$$\varphi_E: (x, \sqrt{1-z^2}, z) \mapsto \varphi_E(x, \sqrt{1-z^2}, z) = (x, z)$$

$$\varphi_E^{-1}: (x, z) \mapsto \varphi_E^{-1}(x, z) = (x, \sqrt{1-z^2}, z)$$