Lecture 21

1. Scalar functions with the manifold as their domain.
2. Coordinate representative of a function.
Example of a manifold with two coordinate charts

Example 2 (Real $n \times n$ non-singular matrices)

Let $M = \{ A \in \mathbb{R}^{n \times n} : \det(A) \neq 0 \}$ be the set of all $n \times n$ real non-singular matrices.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the determinant function $\det: A \to \mathbb{R}$.

Let $\mathbb{R}^n$ be the set of all matrices having non-zero determinant.

$M$ is a manifold.

(i) The chart $(\phi, U)$ with

$\phi: U = M \to \mathbb{R}^n$

and $\phi(A) = (\phi_1(A), \phi_2(A), \ldots, \phi_n(A))$

is a globally defined coordinate chart. ("system")

(ii) The chart $(\bar{\phi}, U)$ with

$\bar{\phi}: U = M \to \mathbb{R}^n$

and $\bar{\phi}(A) = (\bar{\phi}_1(A), \bar{\phi}_2(A), \ldots)$

($\text{Remark: } \bar{\phi}(U) = \mathbb{R}^n$)

is another coordinate system.

(iii) These two charts are $C^\infty$ related because the transition map $\bar{\phi} \circ \phi^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is:

$$(a_1, a_2, \ldots) \mapsto \bar{\phi} \circ \phi^{-1}(a_1, a_2, \ldots) = (a_{12}, a_{13}, \ldots)$$

is infinitely differentiable. Indeed its Jacobian $J_\phi(\bar{\phi} \circ \phi^{-1})$: 

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
\end{bmatrix}
$$
A key attribute of a manifold, say \( M \), is that it serves as the domain of real valued functions. Let us develop their definition.

**Definition (\( C^s \) function)**

a) Let \( M \) be a manifold. A real valued function \( f : M \to \mathbb{R}^2 \) is said to be a **\( C^s \) function** (\( s \leq r \)), denoted by \( f \in C^s(M, \mathbb{R}^2) \), if for each chart \((\phi, U)\) the "coordinate representative"

\[
f_{\text{rep}}(x^1, \ldots, x^n) = f \circ \phi^{-1}(x^1, \ldots, x^n)
\]

has partial derivatives which are continuous up to order \( s \).

Please note that this definition is the non-linear analogue of the coordinate representative introduce on Page 2012 in the context of linear mathematics.

b) A real valued function \( f \) is said to be "\( C^s \) in a neighborhood of \( \phi \)" denoted by \( f \in C^s(M, \mathbb{R}^2) \) if

\[
U = (\text{domain of } \phi) \subset M
\]

is an open set containing \( \phi \), and \( f \in C^s(U, \mathbb{R}^2) \).
Comments:
We are able to define $C^\infty$ functions on $M$ because (a) $M$ looks locally like $\mathbb{R}^n$ and we know about $C^\infty$ functions on $\mathbb{R}^n$.

(b) if

$$U_1 = \text{domain } \varphi_1, \quad U_1 \cap U_2 \neq \emptyset, \quad U_2 = \text{domain } \varphi_2$$

then the concept of $C^\infty$ functions on $U_1 \cap U_2$ is the same relative to $\varphi_1$, as it is relative to $\varphi_2$ because $\varphi_1 \circ \varphi_2^{-1}$ is a $C^r$ homeomorphism.

Frame work surrounding an $n$-dimensional manifold are the same as those surrounding an $n$-dimensional vector space on page 2014.
Example 3 (Unit 2-sphere $S^2$)

Let $M = S^2$.

Define three real valued functions $x^1, x^2, x^3$ and let $M = S^2$.

For any $f$ defined on $U_N$, we have:

For any $f$ defined on $U_E$, we have:

3.) Transition map:

$\phi_N: (x, y, z) \rightarrow (x, y, 1 - x^2 - y^2) = (x, y, z)$

$\phi_N^{-1}: (x, y, z) \rightarrow (x, y, 1 - x^2 - y^2) = (x, y, z)$

$\phi_E: (x, y, z) \rightarrow (x, y, 1 - x^2 - y^2) = (x, y, z)$

$\phi_E^{-1}: (x, y, z) \rightarrow (x, y, 1 - x^2 - y^2) = (x, y, z)$