

## Lecture 22

Coordinate functions of a chart

Tangent vector and the transformation  
of its components under a change  
of coordinate charts.

Vector as a derivation

[ Singer & Thorpe §5.1; P97-101  
Hicks chapter 1 ]

22.1

an  $n$ -dimensional

Given a  $C^\infty$  manifold, say  $M$ , we shall find that the coordinate charts (= systems) of  $M$  give rise to pointwise defined tangent vectors, which are nothing more and nothing less than directional derivatives, i.e. derivations that act on functions ( $\in C^\infty(M, \mathbb{R})$ ) at any chosen point  $P$  in  $M$ . As the word "vector" implies, the tangent vector at a point  $P$  is a coordinate invariant concept; i.e. it is independent of the coordinate chart relative to which that vector is constructed.

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Every coordinate chart of an  $n$ -dimensional manifold has  $n$  coordinate functions.

They play a fundamental role in the formation of a tangent vector and they are defined as follows

Definition (Coordinate functions)

Let  $M$  be an  $n$ -dimensional  $C^\infty$  manifold,

b)  $(\varphi, U)$  be a chart, and let

c)  $x^j : \mathbb{R}^n \rightarrow \mathbb{R}$  be the  $j^{\text{th}}$  coordinate function on  $\mathbb{R}^n$ , namely

$$(a^1, \dots, a^n) \mapsto x^j(a^1, \dots, a^n) = a^j$$

Then  $x^j = x^j \circ \varphi : U \rightarrow \mathbb{R}$

$$P \mapsto x^j(P) = x^j \circ \varphi(P)$$

is the  $j^{\text{th}}$  coordinate function of  $(\varphi, U)$

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Note that each coordinate function  $x^i$

is a  $C^r$  function:

$$x^i \in C^r(U, \mathbb{R})$$

Furthermore,

$$(x^1, x^2, \dots, x^n) = \varphi \in C^r(U, \mathbb{R}^n)$$

is a coordinate system explicitly exhibited in terms of its components.

### ~~Tangent Vector at a Point of the Manifold.~~

~~The concept of a tangent vector at a point is the mathematical generalization of the change of a property (pressure, density, temperature; i.e. anything physically measurable quantity)~~

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### Tangent Vector at a Point of the Manifold

The tangent vector at a point is a concept that refers to the change in any given physical property (e.g. pressure, density, temperature, any  $f \in C^1(U, \mathbb{R})$ ) in relation to the change in position relative to any given coordinate system.

The remarkable thing about the tangent vector is that it unites into a single entity the relation between (a) the changes in a property and (b) the position changes regardless of (a) what these properties are and of

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(b) the coordinate systems used to specify the position changes.

The mathematical concept which mathematizes<sup>(\*)</sup> the causal relation between them (namely, the change in any given property and the position change expressed relative to any given coordinate system) is the directional derivative combined with the multivariable chain rule.

(\*) Notation: "mathematize" = cast into mathematical form.

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This mathematization is achieved by the following

Definition (Tangent vector as a derivation)

Let  $M$  be a manifold and let  $P \in M$  be point in the manifold

Then a tangent vector at  $P$  is a map

$$V: C^\infty(M, \mathbb{R}) \rightarrow \mathbb{R} \text{ i.e. } \left\{ \begin{array}{l} \text{smooth} \\ \text{function} \end{array} \right\} \rightarrow \text{reals}$$

with the following property:

If  $(\varphi, U)$  is a coordinate chart (i.e. system) and

$$P \in U (= \text{domain of } \varphi)$$

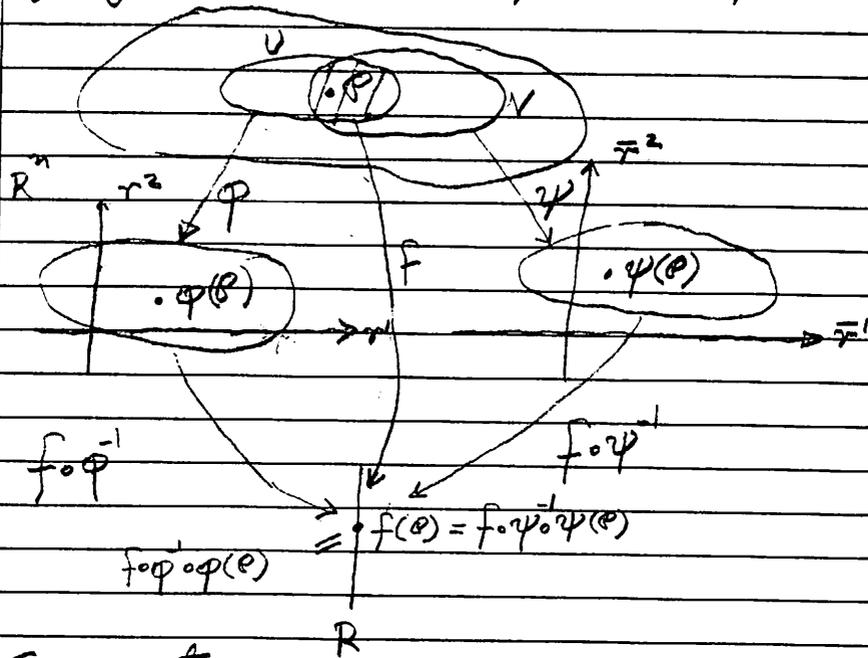
then  $\exists$  exists  $(\alpha^1, \alpha^2, \dots, \alpha^n) \in \mathbb{R}^n$  with the property

$$f \xrightarrow{V} V(f) = \sum_{i=1}^n \alpha^i \left. \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \right|_{\varphi(P)}$$

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Reminder:

The framework of concepts underlying this definition is the same as that underlying the definition of a manifold



Comment:

1. The numbers  $\{\alpha^i\}$  are determined by the set of function  $C^0(M, \mathbb{R})$ .
2. As we shall see, once the numbers  $\{\alpha^i\}$  have been determined relative to one coordinate system, they are known relative to all

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(overlapping) coordinate systems

3. The validity of this definition is established by the following computation.

a) On one hand one has

$$V(f) = \sum_i \alpha^i \frac{\partial (f \circ \phi^{-1})}{\partial r^i} \quad \left| \begin{array}{l} \phi(p) = (r_0^1, \dots, r_0^n) \\ (r_0^1, \dots, r_0^n) \end{array} \right.$$

$$= \sum_{i=1}^n \alpha^i \frac{\partial f \circ \psi \circ \phi^{-1} (r_0^1, \dots, r_0^n)}{\partial r^i} \quad \left| \begin{array}{l} (r_0^1, \dots, r_0^n) \\ (r_0^1, \dots, r_0^n) \end{array} \right.$$

By picking  $n$  different functions  $f$ , one obtains  $\{\alpha^i\}$  if one knows  $V(f)$  for these  $n$  different functions. Conversely, given  $\{\alpha^i\}$  and a choice of a coordinate system,

$V(f)$  is determined  $\forall f$ .

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b) on the other hand, using the chain rule one has

$$V(f) = \sum_{i=1}^n \alpha^i \frac{\partial (f \circ \bar{\varphi}^{-1})}{\partial \bar{r}^i} \Big|_{\varphi(p)} \quad (*)$$

$$= \sum_{i=1}^n \alpha^i \frac{\partial (f \circ \psi^{-1} \circ \psi \circ \bar{\varphi}^{-1})}{\partial \bar{r}^i} \Big|_{\varphi(p)} \quad (+)$$

Recall the following explicit expressions:

$$f \circ \psi^{-1} = f_{\psi \text{ rep}}(\bar{r}^1, \dots, \bar{r}^n) \quad (1)$$

$$\psi \circ \bar{\varphi}^{-1} = \{\bar{r}^1(r^1, \dots, r^n), \dots, \bar{r}^n(r^1, \dots, r^n)\}$$

$$\bar{r}^j \circ \psi \circ \bar{\varphi}^{-1} = \bar{r}^j(r^1, \dots, r^n) \quad (2)$$

Inserting them into Eq (+) we have

chain rule

$$V(f) = \sum_{i=1}^n \alpha^i \frac{\partial f_{\psi \text{ rep}}(\bar{r}^1(r^1, \dots, r^n), \dots, \bar{r}^n(r^1, \dots, r^n))}{\partial \bar{r}^i} \Big|_{\varphi(p)}$$

$$= \sum_{i=1}^n \alpha^i \sum_{j=1}^n \frac{\partial \bar{r}^j}{\partial r^i} \frac{\partial f_{\psi \text{ rep}}(\bar{r}^1, \dots, \bar{r}^n)}{\partial \bar{r}^j} \Big|_{\varphi(p)}$$

defn (1) & (2)

$$= \sum_{i=1}^n \alpha^i \sum_{j=1}^n \frac{\partial (\bar{r}^j \circ \psi \circ \bar{\varphi}^{-1})}{\partial r^i} \frac{\partial f \circ \psi^{-1}}{\partial \bar{r}^j} \Big|_{\varphi(p)}$$

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Introducing the Jacobian matrix for the transition function  $\psi \circ \bar{\varphi}^{-1}$

$$J^j_i(\psi \circ \bar{\varphi}^{-1}) = \frac{\partial \bar{r}^j \circ \psi \circ \bar{\varphi}^{-1}}{\partial r^i} \quad (*)$$

and reverting to the Einstein summation convention, one obtains

$$V(f) = \beta^j \frac{\partial (f \circ \psi^{-1})}{\partial \bar{r}^j} \Big|_{\psi(p) = (\bar{r}^1, \dots, \bar{r}^n)} \quad (**)$$

Here

$$\beta^j = J^j_i(\psi \circ \bar{\varphi}^{-1}) \Big|_{\varphi(p) = (r^1, \dots, r^n)} \alpha^i \quad (***)$$

c) Eqs. (\*) and (\*\*) on page 20.9 and 20.10 are computations using two different sets of coordinate functions, namely  $\{x^i = r^i \circ \varphi\}$  and  $\{\bar{x}^j = \bar{r}^j \circ \psi\}$ .

In light of the chain rule, these computations refer to one and the same tangent

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vector whenever its components  $\{\alpha^i\}$   
and  $\{\beta^i\}$  are related means of the  
Jacobian matrix, Eq. (\*) p20.10, of the  
transition map  $\psi \circ \phi^{-1}$ .

22, 12a

## The Concept of a Tangent Vector:

### A Product of Mental Integration

The mapping  $V$  as defined by the  
computation process on page 20.8-20.10

applies to all functions  $f \in C^1(M, \mathbb{R})$

and holds by means of the chain rule

for all overlapping coordinate charts

which contain the point  $P$ .

INSERT (see p 20.12b; one may skip this on a first reading)

By suppressing reference to any particular  
function and any particular coordinate

chart (with the understanding

that  $V$  can be applied to any function

(GO TO PAGE 22, 23)

BEGIN

INSERT (for P 22.12a)

22, 12b

Nota Bene:

In the theory of Concept-Formation

(Chapter 2, excerpted from "Introduction

to Objectivist Epistemology"), the

fact that the functions  $f$  have the

property common to all elements of

$C^A(M, P, R)$  is referred to as such functions

having a commensurable characteristic,

which is called the CCD (Conceptual

Common Denominator) of these functions.

Similarly, the fact that the overlapping

coordinate charts containing the common

point  $P$  have the property of being

22, 12c

$C^A$ -related (see Page 20.17) is referred

to as such charts having a commensurable

characteristic, which is called their

CCD (Conceptual Common Denominator)

By taking the partial derivatives of

the transition maps between such charts,

one obtains another CCD, this one

of the set of  $n$ -tuples such as  $\{\alpha^i\}$  and

$\{\beta^j\}$ , which are related by Jacobians

matrices  $[J^j_i(\psi \circ \phi^{-1})]$  between them.

The identification of a CCD is the

starting point for the mental process

of forming a new concept. This fact

is explicated in the excerpted Chapter 2

mentioned above. END of INSERT; (Go back to page 22.12a)

(cont'd from 22.12a)

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relative to any coordinate chart,  
but that it must be done so to a  
particular function relative to a  
particular coordinate chart in a  
particular instance)

one has created the concept of a  
tangent vector  $v$

In order to be able to reduce this  
concepts as quickly as possible to the  
things it refers to, one introduces  
with  
 $x^i = r^{\alpha} \circ \varphi$  and  $\bar{x}^{\beta} = \bar{r}^{\beta} \circ \psi$   
the following notation:

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$$V(f) = \alpha^i \frac{\partial (f \circ \varphi)}{\partial r^i} \Big|_{\varphi(p)} \equiv \alpha^i \frac{\partial (f)}{\partial x^i}$$

$$V(f) = \beta^{\beta} \frac{\partial (f \circ \psi)}{\partial \bar{r}^{\beta}} \Big|_{\psi(p)} = \beta^{\beta} \frac{\partial (f)}{\partial \bar{x}^{\beta}}$$

where

$$\beta^{\beta} = J_{\beta}^{\alpha}(\psi \circ \varphi^{-1}) \alpha^i \equiv \frac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}} \alpha^i$$

This holds  $\forall f \in C^0(M, \mathbb{R})$ . Consequently,

$$V = \alpha^i \frac{\partial}{\partial x^i} = \beta^{\beta} \frac{\partial}{\partial \bar{x}^{\beta}} = \alpha^i \frac{\partial \bar{x}^{\beta}}{\partial x^i} \frac{\partial}{\partial \bar{x}^{\beta}}$$

This notation highlights not only the fact that the  
tangent vector  $v$  at a point is a derivation,  
but also how to use it in going from one  
coordinate chart to another.

Recall the definition of a vector. 2.2.15  
From lectures

A set of numbers  $\{\alpha^i\}$  form the components of a vector if the transformation from one frame to another is expressed by a linear transformation

$$\beta^j = \sum_{i=1}^n J^j_i \alpha^i$$

A vector is an equivalence class of  $n$ -tuples in which two  $n$ -tuples are called equivalent if they can be related by a non-singular transformation

$$\{\beta^j\} \equiv \{\alpha^i\} \pmod{GL_n(\mathbb{R})}$$

$\Leftrightarrow \exists J^j_i \in GL_n(\mathbb{R})$  such that

$$\beta^j = \sum_{i=1}^n J^j_i \alpha^i$$

This equivalence class is new entity. It is viewed and used in a specific way. It is a product resulting from a mental integration of two or more units (such as  $\{\alpha^i\}$  and  $\{\beta^j\}$ ) which are isolated according to a specific

2.2.16

characteristic(s) (here, related by an element of  $GL_n(\mathbb{R})$ ) and united by means of a specific definition (like the one above).

The new entity is a vector which we shall designate by the symbol  $V$ .

If elements of  $GL_n(\mathbb{R})$  are the Jacobians  $J^j_i(\psi \circ \varphi^{-1})$  of the transition maps

between overlapping coordinate

charts of  $M$ , then this narrowing process results in a kind of vector

which is called a tangent vector at  $\mathcal{P}$ .

It is designated by

$$V = \alpha^i \frac{\partial}{\partial x^i} = \beta^j \frac{\partial}{\partial x^j}$$

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and its defining property is that it does not single out any preferred coordinate system. This is so because the mapping property

$$f \circ \psi = \psi \circ f$$

on page 22.14 can be expressed in the same way relative to all overlapping coordinate systems  $(\varphi, U)$  and  $(\psi, V)$ .

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Re summarization of P. 22.12b-22.12c

Let us resummariize the mental process leading to the concept tangent vector. Such a vector is a classification of directional derivatives in relation to different coordinate systems.

First one forms the concept of a directional derivative operator. One does this by mentally isolating those linear combinations of partial derivatives of different functions on  $\mathbb{R}^n$  which have the same  $n$ -tuple of coefficients (and thus differ from other linear combinations). Then, by omitting explicit reference to the difference between these functions, one mentally unites these derivatives into a new mental unit, the directional derivative operator characterized by that  $n$ -tuple of coefficients,

$$a^i \frac{\partial}{\partial x^i} : f_{\text{reg}} \mapsto a^i \frac{\partial f_{\text{reg}}}{\partial x^i}$$

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Next one forms the concept of a tangent vector. One does this by considering the directional derivative operators relative to different coordinate systems. This means that the directional derivative operators differ from one another not only in that they are characterized by different  $n$ -tuples, but also in that the partial derivatives are induced by different coordinate systems.

Being linear combinations of partial derivative operators, the directional derivative operators differ from one another in that (i) each is characterized by a different  $n$ -tuple, and (ii) the partial derivatives are different because they come from different coordinate systems.

one forms the concept of a tangent vector by isolating those linear combinations of partial derivatives, say,

$$a^i \frac{\partial}{\partial r^i}, \quad b^{j'} \frac{\partial}{\partial r^{j'}}, \quad c^{k''} \frac{\partial}{\partial r^{k''}}, \text{ etc.}$$

which differ from another by

$$\frac{\partial}{\partial r^{j'}} = \Lambda^i_{j'} \frac{\partial}{\partial r^i}, \quad \frac{\partial}{\partial r^{k''}} = \Lambda^i_{k''} \frac{\partial}{\partial r^i}, \text{ etc}$$

$$b^{j'} = \Lambda^{j'}_l a^l, \quad c^{k''} = \Lambda^{k''}_l a^l, \text{ etc}$$

but have the common distinguishing feature that

$$\Lambda^i_{j'} \Lambda^{j'}_l = \Lambda^i_{k''} \Lambda^{k''}_l = \dots = \delta^i_l$$

By omitting explicit reference to the difference in such linear combinations, i.e. viewing them as a single equivalence class, one mentally unites them into a new mental unit, the tangent vector

$$a^i \frac{\partial}{\partial r^i} = b^{j'} \frac{\partial}{\partial r^{j'}} = c^{k''} \frac{\partial}{\partial r^{k''}} = \dots \text{ etc}$$