Lecture 22

Coordinate functions of a chart

Tangent vector and the transformation of its components under a change of coordinate charts.

Vector as a derivation

[Singer & Thorpe §5.1; P97-101
Hicks Chapter 1]
Given a $C^\infty$ manifold, say $M$, we shall find that the coordinate charts (= systems) of $M$ give rise to pointwise defined tangent vectors, which are nothing more and nothing less than directional derivatives, i.e., derivations that act on functions $f \in C^\infty(M, \mathbb{R})$ at any chosen point $p$ in $M$. As the word "vector" implies, the tangent vector at a point $p$ is a coordinate invariant concept, i.e., it is independent of the coordinate chart relative to which that vector is constructed.

Every coordinate chart of an $n$-dimensional manifold has $n$ coordinate functions. They play a fundamental role in the formation of a tangent vector and they are defined as follows.

**Definition (Coordinate function)**

Let $M$ be an $n$-dimensional $C^\infty$-manifold, $\psi(p, U)$ be a chart, and let $\tilde{\xi} : \mathbb{R}^n \to \mathbb{R}$ be the $j^{th}$ coordinate function on $\mathbb{R}^n$, namely

$$ (a_1, \ldots, a_n) \mapsto \tilde{\xi}(a_1, \ldots, a_n) = a_j. $$

Then $x^j = \tilde{\xi} \circ \psi : U \to \mathbb{R}$

$$ p \mapsto x^j(p) = \tilde{\xi} \circ \psi(p) $$

is the $j^{th}$ coordinate function of $(\psi, U)$. 
Note that each coordinate function $x^i$ is a $C^n$ function:

$$x^i \in C^n(U, \mathbb{R})$$

Furthermore,

$$(x'_1, x'_2, \ldots, x'_n) = \Phi \in C^n(U, \mathbb{R}^n)$$

is a coordinate system explicitly exhibited in terms of its components.

**Tangent Vector at a Point of the Manifold**

The tangent vector at a point is a concept that refers to the change in any given physical property (e.g., pressure, density, temperature, any $f \in C^m(U, \mathbb{R})$) in relation to the change in position relative to any given coordinate system.

**Tangent Vector at a Point of the Manifold**

The remarkable thing about the tangent vector is that it unifies into a single entity, the relation between (a) the change in a property and (b) the position changes regardless of (c) what these properties are and of
(b) the coordinate systems used to specify
the position changes.

The mathematical concept which mathematizes
the causal relation between them
(namely, the change in any given property
and the position change expressed relative
to any given coordinate system) is the
directional derivative combined with
the multivariable chain rule.

This mathematization is achieved by
the following

Definition (Tangent vector as a derivation)

Let $M$ be a manifold and let $p \in M$ be a point in the manifold.

Then a tangent vector at $p$ is a map

$$\nabla : C^\infty(M, \mathbb{R}) \to \mathbb{R}$$

with the following property:

If $(p, U)$ is a coordinate chart (i.e., system)
and $p \in U$ (domain of $\phi$),

then $\exists$ exists $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ with the
property

$$f \mapsto \nabla(f) = \sum_{i=1}^n x^i \frac{\partial (f \circ \phi)}{\partial x^i} \bigg|_{\phi(p)}$$

*Nota bene: "mathematize" = cast into mathematical form.
Reminder:
The framework of concepts underlying this definition is the same as that underlying the definition of a manifold.

\[ \phi \circ \Psi = f \circ \phi \circ \psi \]

Comment:
1. The numbers \( \xi^a_i \) are determined by the set of function \( C^0(M, \mathbb{R}) \).
2. As we shall see, once the numbers \( \xi^a_i \) have been determined relative to one coordinate system, they are known relative to all overlapping coordinate systems.

3. The validity of this definition is established by the following computation.

\[ V(f) = \sum_{i=1}^n \xi^i \frac{\partial f \circ \phi \circ \psi}{\partial \tau^i} |_{\phi(\psi(\mathfrak{p})))} \]

By picking \( n \) different functions \( f \), one obtains \( \xi^a_i \) if one knows \( V(f) \) for these \( n \) different functions. Conversely, given \( \xi^a_i \) and a choice of a coordinate system, \( V(f) \) is determined for all \( f \).
b) on the other hand, using the chain rule one has

\[ V(f) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \phi_i} \phi_i \]  

(1)

\[ = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \phi_i} \psi \phi \]  

(2)

Recall the following explicit expressions:

\[ \psi \phi = \{ \phi_1, \phi_2, \ldots, \phi_m \} \]

\[ \phi_i \psi \phi = \frac{\partial \phi_i}{\partial \phi_j} \psi \phi \]

Inserting them into Eq.(1) we have

\[ V(f) = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial \phi_i} \phi_i \]  

(3)

Introducing the Jacobian matrix

\[ J_\psi \phi \psi^{-1} \]  

(4)

and reverting to the Einstein summation convention, one obtains

\[ V(f) = \beta_\phi \frac{\partial}{\partial \phi} \psi \phi \]  

(5)

Here

\[ \beta_\phi = \frac{\partial \psi \phi^{-1}}{\partial \psi} \]

(6)

Equations (1) and (5) on page 22.9 and 22.10 are computations using two different sets of coordinate functions, namely

\[ \{ x^i = \phi \psi \} \]  

and \[ \{ x^i = \phi \psi \} \].

In light of the chain rule, these computations refer to one and the same tangent
vector whenever its components \( x^i \)

and \( \{ \beta^i \} \) are related means of the

Jacobian matrix, Eq. (1) p20.12 of the

transition map \( \Psi \circ \Phi^t \).
The Concept of a Tangent Vector:

A Product of Mental Integration

The mapping \( \mathbf{V} \) as defined by the computation process on page 20.8-20.10 applies to all functions \( f \in \mathcal{C}^1(M, \mathbb{R}) \) and holds by means of the chain rule for all overlapping coordinate charts, which contain the point \( P \).

By suppressing reference to particular functions and particular coordinate chart (with the understanding that \( \mathbf{V} \) can be applied to any function).

(GO TO PAGE 22.23)
Nota Bene:

In the Theory of Concept-Formation (Chapter 2, excerpted from "Introduction to Objectivist Epistemology"), the fact that the functions f have the property common to all elements of C[^a](M,ν,R) is referred to as such functions having a commensurable characteristic, which is called the CCD (Conceptual Common Denominator). By taking the partial derivatives of the transition maps between such charts one obtains another CCD, this one of the set of n-tuples such as [φ] and [φ'], which are related by Jacobians of matrices [J[^a]((ψ*φ'))] between them. The identification of a CCD is the starting point for the mental process of forming a new concept. This fact is explicated in the excerpted Chapter 2 mentioned above.
relative to any coordinate chart, but that it must be done so to a particular function relative to a particular coordinate chart in a particular instance. One has created the concept of a tangent vector $\mathbf{V}$.

In order to be able to reduce this concepts as quickly as possible to the things it refers to, one introduces with $\alpha^i = \gamma^i \varphi$ and $\alpha^i = \Gamma^i_{jk} \psi$

the following notation:

\[
V(f) = \alpha^i \frac{\partial (f \circ \varphi)}{\partial x^i \varphi} = \alpha^i \frac{\partial f}{\partial x^i} \varphi \quad \psi(f) = \beta^i \frac{\partial (f \circ \varphi)}{\partial x^i \varphi} = \beta^i \frac{\partial f}{\partial x^i} \psi
\]

where

\[
\beta^i = \int \varphi^{-1} ( \psi \cdot \varphi ) \quad \alpha^i = \frac{\partial (x^i \varphi)}{\partial x^i} \alpha^i
\]

This holds $\forall f \in C^1(M, \mathbb{R})$. Consequently,

\[
V = \alpha^i \frac{\partial}{\partial x^i} = \beta^i \frac{\partial}{\partial x^i} = \alpha^i \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j} \Delta^2
\]

This notation highlights not only the fact that the tangent vector $\mathbf{V}$ at a point is a derivation, but also how to use it in going from one coordinate chart to another.
Recall the definition of a vector. A set of numbers \( \{ \alpha^i \} \) form the components of a vector if the transformation from one frame to another is expressed by a linear transformation

\[
\beta^i = \sum_{i=1}^{n} J^i_j \alpha^j
\]

A vector is an equivalence class of \( n \)-tuples in which two \( n \)-tuples are called equivalent if they can be related by a non-singular transformation

\[
\{ \beta^i \} \equiv \{ \alpha^i \} \mod GL_n(\mathbb{R})
\]

\[
\iff \exists J^i_j \in GL_n(\mathbb{R}) \text{ such that } \beta^i = \sum_{i=1}^{n} J^i_j \alpha^j
\]

This equivalence class is a new entity. It is viewed as used in a specific way. It is a product resulting from a mental integration of two or more units (such as \( \alpha^1 \) and \( \alpha^3 \)) which are isolated according to a specific

characteristic(s) (here, related by an element of \( GL_n(\mathbb{R}) \) and united by means of a specific definition (like the one above).

The new entity is a vector which we shall designate by the symbol \( \mathbf{v} \).

If elements of \( GL_n(\mathbb{R}) \) are the Jacobians \( J^i_j(\Psi, \Phi) \) of the transition maps \( \Phi(\Psi) \) between overlapping coordinate charts of \( M \), then this narrowing process results in a kind of vector which is called a tangent vector at \( \mathbf{p} \).

It is designated by

\[
\mathbf{v} = \alpha^i \frac{\partial}{\partial x^i} = \beta^i \frac{\partial}{\partial x^i}
\]
and its defining property is that it does not single out any preferred coordinate system. This is so because the mapping property
\[ f^* \mathbf{v} = \mathbf{v}(f) \]
on page 22.14 can be expressed in the same way relative to all overlapping coordinate systems \((\varphi, U)\) and \((\psi, V)\).
Two summarise the mental process leading to the concept tangent vector, such a vector is a classification of directional derivatives in relation to different coordinate systems.

Next one forms the concept of a tangent vector. One does this by considering the directional derivative operators relative to different coordinate systems. This means that the directional derivative operators differ from one another not only in that they are characterized by different \( n \)-tuples, but also in that the partial derivatives are induced by different coordinate systems.

Being linear combinations of partial derivative operators, the directional derivative operators differ from one another in that (i) each is characterized by a different \( n \)-tuple, and (ii) the partial derivatives are different because they come from different coordinate systems.
one forms the concept of a tangent vector by isolating these linear combinations of partial derivatives, say
\( a^i \frac{\partial}{\partial r^i} \), \( b^j \frac{\partial}{\partial r^j} \), \( c^k \frac{\partial}{\partial r^k} \), etc., which differ from another by
\( \frac{\partial}{\partial r^j} = \lambda^i_j \frac{\partial}{\partial r^i} \), \( \frac{\partial}{\partial \xi^n} = \lambda^i_n \frac{\partial}{\partial r^i} \), etc.,
\( b^i = \lambda^i \cdot a^i \), \( c^i = \lambda^i \cdot a^i \), etc., but have the distinguishing feature that
\( \lambda^i \cdot \lambda^i = \lambda^i \cdot \lambda^i = \lambda^i \cdot \lambda^i = \cdots = \delta^i \)
By omitting explicit reference to the difference in such linear combinations, i.e. viewing them as a single equivalence class, one mentally unites them into a new mental unit, the tangent vector
\( a^i \frac{\partial}{\partial r^i} = b^j \frac{\partial}{\partial r^j} = c^k \frac{\partial}{\partial \xi^n} = \cdots \) etc.