Tangent to a curve;
[ Singer & Thorpe 55,1; Hicks Ch.2]

Supplement to Lecture 24

Example: The Rotation Group SO(3)
as a manifold with three
independent nowhere-zero vector
fields [MTW Problem 9.13]

Integral curves of a vector field
[S&T, 7.125-126]

[ Vector field induced 1-parameter group
  of transformations. ]
\[ \langle \sigma : \chi \rangle = \sigma (\chi) \]
\[ \langle \omega_i : \varepsilon_i \rangle = \delta^i_j \]

\[ \text{linear} \]
\[ \text{algebra} \]

\[ \lim_{\text{algebra}} \left( \frac{d f}{d x^i} \right) = \frac{\partial f}{\partial x^i} \]
\[ \left( \frac{d x^i}{\partial x^j} \right) \frac{\partial x^j}{\partial x^i} = \delta^i_j \]

\[ \text{calculated} \]
\[ \langle \sigma_i (x) \frac{d x^i}{d x^j} \rangle \frac{d x^j}{d x^i} = 0 \]
\[ \langle \sigma_i (x) \frac{d x^i}{d x^j} \rangle = \langle x \rangle \langle x \rangle \]

\[ \langle \frac{d x^i}{d x^j} \rangle = \delta^i_j \]
24.1

Vector tangent to a curve

The definition of a vector as a derivative arises quite naturally when one considers a curve passing through the sequence of level surfaces of a function.

Consider a curve \( c(t) \) through a point \( P \)

\[ c : \mathbb{R} \rightarrow M \]

\[ t \mapsto c(t) \]

with coordinate representative

\[ t \mapsto \varphi \circ c(t) = \varphi(c(t)) = \{c_1(t), \ldots, c_n(t)\} \]

24.2

The tangent to \( c(\tau) \) at \( P \) is obtained from the given curve \( c(\tau) \) as follows.

**Definition (Tangent to a curve)**

The tangent to \( C \) at \( P_0 = c(\tau) \) is the map

\[ U : C^\infty(M, P_0, \mathbb{R}) \rightarrow \mathbb{R}^1 \]

\[ f \mapsto U(f) = \left. \frac{d}{d\tau} \right|_{\tau_0} f \circ c(\tau) \]

\[ = \left. \frac{d}{d\tau} \right|_{\tau_0} \varphi \circ \varphi \circ c(\tau) \]

\[ = \left. \frac{\partial f}{\partial x_i} \right|_{\varphi} \frac{dc_i(\tau)}{d\tau} \]

\[ = \frac{dc_i}{d\tau} \left. \frac{\partial f}{\partial x_i} \right|_{\varphi} \]

Thus we have the conclusion

\[ U = \frac{dc_i}{d\tau} \left. \frac{\partial f}{\partial x_i} \right|_{\varphi} \]

(\( \mathbf{4} \tau \))

One readily sees that \( U \) is a derivation, i.e., \( U \) is a vector which is determined by the curve \( c(\tau) \).
Differentiation applied to a curve yields its tangent vectors, which comprise a vector field. Integration starts with a given vector field and tries to reconstruct the curves whose tangents are the given vectors.

Vector Field as a Flow

For an appropriate vector field, the following theorem guarantees the existence of a unique curve passing through a given point. The application of this theorem to the points in a coordinate neighborhood yields a 1-parameter group of transformations.
Definition: Let \( \mathbf{U} \) be a smooth vector field on \( M \).

An integral curve of \( \mathbf{U} \) is a smooth curve \( \gamma(t): (a,b) \to M \) such that the tangent vector to \( \gamma \) at each point is one of the assigned vectors of \( \mathbf{U} \) at \( \gamma(t) \):

\[
\dot{\gamma}(t) = \mathbf{U}(\gamma(t)) \quad \text{for } a < t < b
\]

or

\[
\frac{d}{dt} \frac{d\gamma^i}{dt} = \mathbf{a}^i(\gamma(t), \ldots, \gamma^n(t)) \frac{d}{dt}
\]

or

\[
\frac{d^2\gamma^i}{dt^2} = \mathbf{a}^i(\gamma(t), \ldots, \gamma^n(t))
\]