

# Lecture ~~24~~ 24

Tangent to a curve;

[Singer & Thorpe §5.1 ; Hicks ch.1]

## Supplement to Lecture 24

Example: The Rotation Group  $SO(3)$  as a manifold with three independent nowhere-zero vector fields [MTW Problem 9.13]

Integral curves of a vector field

[S&T, §125-126]

[Vector field induced 1-parameter group of transformations.

24.0

Reminder and Review of Results  
of Lecture 23

$$\langle \sigma; v \rangle = \sigma(v)$$

$$\langle \omega^i, e_i \rangle = \delta^i{}_i$$

} linear  
algebra

lim.  
algebra

$$\left\langle df, \frac{\partial}{\partial x^i} \right\rangle = \frac{\partial f}{\partial x^i}$$

+  
calculus

$$\left\langle dx^i, \frac{\partial}{\partial x^i} \right\rangle = \frac{\partial x^i}{\partial x^i} = \delta^i{}_i$$

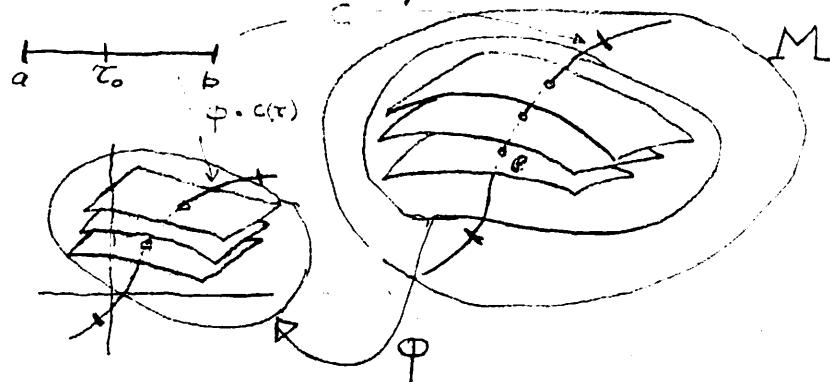
$$\left\langle \sigma_j(x) dx^i, v^i(x) \frac{\partial}{\partial x^i} \right\rangle = \underbrace{\sigma_j v^i}_{\delta^j{}_i} \left\langle dx^i, \frac{\partial}{\partial x^i} \right\rangle = \underbrace{\sigma_j v^i}_{\delta^j{}_i}$$

-6-

vector tangent to a curve

24.1

IV The definition of a vector as a derivative arises quite naturally when one considers a curve passing through the sequence of level surfaces of a function



Consider a curve  $c(\tau)$  through a point  $P$

$$c : \mathbb{R}^1 \rightarrow M$$

$$\tau \mapsto c(\tau)$$

with coordinate representative

$$\tau \mapsto \varphi \cdot c(\tau) \equiv \varphi(c(\tau)) = \{c^1(\tau), \dots, c^n(\tau)\}$$

-7-

24.2

The tangent to  $c(\tau)$  at  $P$  is obtained from the given curve  $c(\tau)$  as follows

Definition (Tangent to a curve)

The tangent to  $c$  at  $P_0 = c(\tau_0)$  is the map

$$\mathcal{U} : C^\infty(M, P_0, \mathbb{R}') \rightarrow \mathbb{R}'$$

$$\begin{aligned} f &\mapsto \mathcal{U}(f) = \frac{d}{d\tau} f \cdot c(\tau) \Big|_{\tau_0} \\ &= \frac{d}{d\tau} f \cdot \underbrace{\varphi \cdot \varphi \cdot c(\tau)}_{\varphi(c)} \Big|_{\tau_0} \\ &= \underbrace{\frac{\partial f}{\partial x^i}}_{\varphi(c)} \Big|_{\tau_0} \frac{dc^i(\tau)}{d\tau} \Big|_{\tau_0} \end{aligned}$$

$$= \frac{dc^i}{d\tau} \frac{\partial f}{\partial x^i}$$

Thus we have the conclusion

$$\boxed{\mathcal{U} = \frac{dc^i}{d\tau} \frac{\partial}{\partial x^i}} \quad \left( = \frac{d}{d\tau} \right)$$

One readily sees that  $\mathcal{U}$  is a derivation i.e. that  $\mathcal{U}$  is a vector which is determined by the curve  $c(\tau)$ .

8a

24.3

Differentiation applied to a curve yields its tangent vectors, which comprise a vector field.

Integration starts with a given vector field and tries to reconstruct the curves whose tangents are the given vectors.

8b

24.4

### Vector Field as a Flow

For an appropriate <sup>given</sup> vector field, the following theorem guarantees the existence of a unique curve passing through a given point.

The application of this theorem to the points in a coordinate neighborhood yields a 1-parameter group of transformations.

8c  
cont'd from p24)

Definition : Let  $\mathbf{U}$  be a smooth vector field on  $M$

24.5

An integral curve of  $\mathbf{U}$  is a smooth

curve  $C(\tau) : (a, b) \rightarrow M$

$\mathbf{U} = a^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}$  such that the tangent vector  
is given to  $C$  at each point is one of the  
assigned vectors of  $\mathbf{U}$  at  $P$ :

$$\dot{C}(\tau) = \mathbf{U}(C(\tau)) \quad \text{for } a \leq \tau < b$$

$$\text{or } \frac{dc^i}{d\tau} \frac{\partial}{\partial x^i} = a^i(C^1(\tau), \dots, C^n(\tau)) \frac{\partial}{\partial x^i}$$

$$\text{or } \boxed{\frac{dc^i}{d\tau} = a^i(C^1(\tau), \dots, C^n(\tau))} \quad i=1, \dots, n$$