Lecture 25

1) Integral curve of a vector field [S&T p125-126]
2) Commutator of two vector fields
   \[ \text{MTW Box 8.4 E, Box 9.2, 89.6 ; S&T, P127} \]
3) The differential (one form) of a function
   \[ \text{MTW 9.4; T. Apostol : Math. Analysis P103-107} \]
What is a Theorem? Or:

Is Deduction Possible without Induction?

Many admirers of mathematics, be they philosophers, engineers, physicists, or even mathematicians themselves, hold mathematics in high regard as a deductive science. However, they are missing the point. Before one has deduction one must have induction. Before one deduces that Socrates is a mortal, one must have induced (ultimately starting from observations) that all men are mortal.

Mathematical theorems are a good example. Before one proves a theorem, i.e., before one deduces certain consequences from the

what is known, given, or assumed, one must assemble, identify precisely, and state explicitly the starting point from which the subsequent consequences follow.

This assembly, identification, and articulation process—the inductive process—is much more difficult than a deduction. This because induction is based on all of one’s relevant knowledge, a person’s whole cognitive landscape.

Thus, because of limitations of time, resources, or even ignorance, perseverance...
and intelligence, instructors quite often skimp on the motivation, i.e. the necessary inductive process which gives rise to the generalization expressed by the theorem. (This is done with the hope that the understanding of the student's curiosity will pick up where the instructor did or had to slack off.)
Theorem: (Existence and uniqueness of a solution to a system of 1st order O.D.E.'s)

Given: Let $W$ be an open set in $\mathbb{R}^n$
Let $w_0 \in W$

Let $x: (\xi, \eta) \in W_0 \longrightarrow W$

$\psi = \{ \psi^i \}_{i=1}^n \subset (\xi, \eta) \times W_0 \longrightarrow W$

such that $\psi = \{ \psi^i \}$ is a solution to

$\frac{dx^i}{dt} = \alpha^i(x^{1}, \ldots, x^{n}, t)$

subject to the initial conditions

$c^i(0) = w^i$

i.e., $c^i(t) = \psi^i(\xi, \eta, \ldots, w^n)$

and $\psi^i(0, w^1, \ldots, w^n) = w^i$ for $i = 1, \ldots, n$

In more compact notation, if one introduces the curve

$c_w(\tau) = \psi(\xi, \eta, w)$

then for $1 \leq i \leq n$ one has

$\frac{d}{d\tau} r^i c_w(\tau) = \alpha^i(r^i c_w(\tau), \ldots, r^n c_w(\tau))$

where for $\tau \in (-\epsilon, \epsilon)$

$\gamma^i c_w(0) = \gamma^i(w)$ ("initial coordinates of $c_w(\tau)$")
Discussion: 25.3

Let us introduce the shorthand notation

\[ \{ \psi_t(z, w, \ldots, w) \} = \psi_{t}(w) \]

Then

\[ \psi_t : W_0 \to W \]

\[ w \mapsto \psi_t(w) \]

is a local parameter group of smooth transformation, i.e., \( \psi_t(w) \) is a unique solution!

(a) \[ \psi_{t_1 + t_2}(w) = \psi_{t_1} (\psi_{t_2}(w)) = \psi_{t_1} \circ \psi_{t_2}(w) \]

(b) \[ \psi_{-t} : w = \psi_t(w) = \psi_{t + t} = \psi_{t} \circ \psi_{-t}(w) = \psi_{t} \circ \psi_{-t}(w) \]

The two boxed equations imply that the \( \psi_t \)'s form a group.
| 25.5 | Remark 1: $\psi_c$ is a local 1-parameter group of transformations. It has the properties:
|      | $\psi_c \circ \psi_c = \psi_{c+c}$
|      | $\psi_c = \psi^{-1}_c$ |
| 25.6 | One introduces $X$, the coordinate functions on the manifold $M$.
|      | $X^c = \gamma^c \circ \phi$. Consequently:
|      | $w^c = \gamma^c \circ \phi(p)$
|      | $w^i = \gamma^i \circ \phi(p)$
|      | These are $\phi$-coordinate values of $p$ and $p$.

**Remark 2:** The proof of this theorem consists of recasting the existence and uniqueness theorem on page 251 in terms of a vector field and curves relative to a coordinate system (chart) $\phi$. In terms of $\phi$, the vector field on $W$ is now denoted by:
|      | $\psi_c = \phi_0(x, \phi(p), \ldots, x^n \phi(p)) \frac{\partial}{\partial x^i} = \phi_z(x^i) \frac{\partial}{\partial x^i}$ |

**Equations** (A) and (B) on page 25.2 become:

**A'** $\frac{d}{dt} \phi(t) = \alpha(t)$

**B'** $x^i \phi(0) = w^i$ (initial $\phi$-coordinate values of the to-be-found curve $\phi(t)$)
25.7

The existence of a unique solution is guaranteed if there exists an open neighborhood of set $W$ such that for each point $p \in U \subseteq W$, there exists a curve with initial point $\theta$ satisfying Eqs. (8) and (8') and $T > 2\pi$. $c(\epsilon) = p(\epsilon)$. $\epsilon = \epsilon(p)$.