

Lecture 25

For Lecture 26

- 1.) Integral curve of a vector field [S&T P125-126]
- 2.) Commutator of two vector fields

[MTW Box 8.4 E, Box 9.2, §9.6 ; S&T. P127]

- 3.) The differential (one form) of a function

[MTW §9.4; T. Apostol : Math. Analysis P103-107]

What is a Theorem? OR: 25.09
Is Deduction Possible without Induction?

Many admirers of mathematics, be they philosophers, engineers, physicists, or even mathematicians themselves, hold mathematics in high regard as a deductive science. However, they are missing the point. Before one has deduction one must have induction. Before one deduces that Socrates is a mortal, one must have induced (ultimately starting from observations) that all men are mortal.

Mathematical theorems are a good example. ^(like the one on P25.1) Before one proves a theorem, i.e., before one deduces certain consequences from the

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what is known, given, or assumed, one must assemble, identify precisely, and state explicitly the starting point from which the subsequent consequences follow.

This assemble, identification, and articulation process - the inductive process - is much more difficult than a deduction. This ^{is} because induction is based on all of one's relevant knowledge, a person's whole cognitive landscape.

Thus, because of limitations of time, resources, or even ignorance, perseverance

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and intelligence, instructors quite often

skimp on the motivation, i.e. the
necessary

inductive process which gives rise to the

generalization expressed by the theorem

(This is done with the hope)

theory, principle, etc. and the under-

standing that the student's curiosity

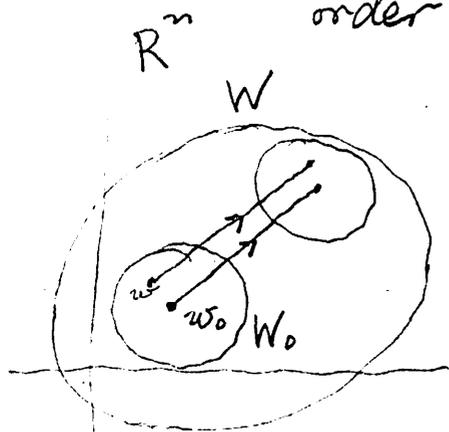
will pick up where the instructor

did (or had to) slack off.

Vector Field as a Flow.

25.1

Theorem: (Existence and uniqueness of a solution to a system of 1st order o.d.e.'s)



Given: Let W be an open set in \mathbb{R}^n
Let $w_0 \in W$

Let $\alpha^i(x^1, \dots, x^n) \in C^\infty(W, \mathbb{R}^1)$ $i=1, \dots, n$

Conclusion: There exists

- (i) an open set $W_0 \subset W$ about w_0
- (ii) an open interval $(-\epsilon, \epsilon) \subset \mathbb{R}^1$
- (iii) a unique smooth n -component map

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25.2

$$\psi = \{\psi^i\}_{i=1}^n : (-\epsilon, \epsilon) \times W_0 \rightarrow W$$

$$(\tau, w^1, \dots, w^n) \mapsto \{\psi^i(\tau, w^1, \dots, w^n)\}$$

such that $\psi = \{\psi^i\}$ is a solution to

$$\frac{dC^i}{d\tau} = \alpha^i(C^1(\tau), \dots, C^n(\tau))$$

subject to the initial conditions

$$C^i(0) = w^i$$

i.e. $C^i(\tau) = \psi^i(\tau, w^1, \dots, w^n)$

and $\psi^i(0, w^1, \dots, w^n) = w^i$ for $i=1, \dots, n$

In more compact notation, if one introduces the curve

$$C_w(\tau) = \psi(\tau, w)$$

then for $1 \leq i \leq n$ one has

A) $\frac{d}{d\tau} r^i \circ C_w(\tau) = \alpha^i(r^1 \circ C_w(\tau), r^2 \circ C_w(\tau), \dots, r^n \circ C_w(\tau))$

where for $\tau \in (-\epsilon, \epsilon)$

B) $r^i \circ C_w(0) = r^i(w)$ ("initial coordinates of $C_w(\tau)$ ")

Discussion: - 11 - 25.3
 Let us introduce the short hand notation

$$\{\psi^t(\tau, w_1, \dots, w^n)\} \equiv \psi_\tau(w)$$

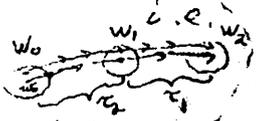
Then

$$\psi_\tau : W_0 \rightarrow W$$

$$w \mapsto \psi_\tau(w)$$

is a local 1-parameter group of smooth transformation, i.e., $\psi_\tau(w)$ is a unique sol'n!

(a) $\psi_{\tau_1 + \tau_2}(w) \stackrel{!}{=} \psi_{\tau_1}(\psi_{\tau_2}(w)) \equiv \psi_{\tau_1} \circ \psi_{\tau_2}(w)$



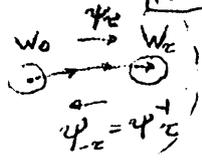
$$\psi_{\tau_1 + \tau_2} = \psi_{\tau_1} \circ \psi_{\tau_2}$$

$\psi_\tau(w)$ is a unique sol'n!

(b) $\psi_{-\tau} : w = \psi_0(w) = \psi_{-\tau + \tau}(w) \stackrel{!}{=} \psi_{-\tau}(\psi_\tau(w))$
 $\equiv \psi_{-\tau} \circ \psi_\tau(w)$

$$\psi_{-\tau} \circ \psi_\tau = \text{identity}$$

$$\psi_{-\tau} = \psi_\tau^{-1}$$



The two boxed equations imply that the ψ_τ 's form a group.

25.4

Theorem (Flow on a Manifold)

Let M be a smooth manifold
 Let V " " " vector field on M
 Let $P_0 \in M$

Then:

- (1) an open set U about P_0
 - (2) an open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$
 - (3) a smooth map $\psi : (-\epsilon, \epsilon) \times U \rightarrow M$
- such that

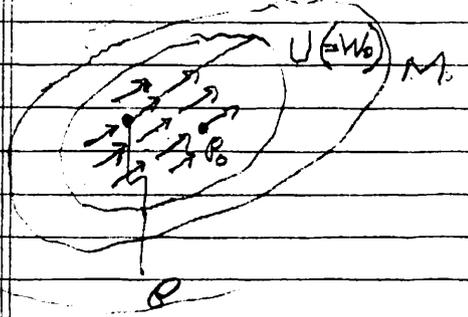
$\forall P \in U$ the curve
 $c_P : (-\epsilon, \epsilon) \rightarrow M$

defined by

$$c_P(\tau) = \psi_{\tau c}(P)$$

error in
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is the unique integral curve
 $(-\epsilon, \epsilon) \rightarrow M$
 of V with
 $c_P(0) = P$



25.5

Remark 1: Ψ_τ is a local 1-parameter group of transformations

It has the properties

$$\Psi_{\tau_1} \circ \Psi_{\tau_2} = \Psi_{\tau_1 + \tau_2}$$

$$\Psi_\tau = \Psi_\tau^{-1}$$

Remark 2: The proof of this theorem consists of recasting the existence and uniqueness theorem on P 25.1 in terms of a vector field and curves relative to a coordinate system (= chart) φ containing $P_0 \in M$.

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on the manifold M

a) One introduces the coordinate functions

$$x^i = r^i \circ \varphi$$

Consequently

$$w^i = r^i \circ \varphi(P_0)$$

$$w^i = r^i \circ \varphi(P)$$

are φ -coordinate values of P_0 and P .

b) In terms of φ , the vector field on W is now denoted by

$$u_\varphi = \alpha^i(r^i \circ \varphi, \dots, r^n \circ \varphi) \frac{\partial}{\partial r^i} = \alpha^i(x^i) \left. \frac{\partial}{\partial x^i} \right|_P$$

defined on $W \subset M$.

c) Equations (A) and (B) on Page 25.2 become:

$$(A') \quad \frac{d c(\tau)}{d\tau} = \alpha^i(\varphi) \Big|_{P=c(\tau)} \cdot \frac{\partial}{\partial x^i}$$

$$(B') \quad x^i(c(0)) = w^i \quad \left(\begin{array}{l} \text{"initial } \varphi\text{-coordinate} \\ \text{values of the} \\ \text{to-be-found curve } c(\tau)\text{"} \end{array} \right)$$

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d) The existence & uniqueness theorem
now guarantees that there exists
a nbhd

$$U = W_0$$

around p_0 such that for each point

$$p \in U = W_0 (\subset W \subset M)$$

the curve with initial point p

$$c_p(\tau) = \psi_\tau(p)$$

$$p \circ c_p(\tau) = \{ \psi^\tau(\tau, w^1, \dots, w^n) \}$$

satisfies Eqs (A') and (B') on p 25.6

