Lecture 27

Differential 1-form as the linear-approximation of a function \([\text{MTW} \S 9.4]\)

\(\text{Diff}'l p\text{-forms}: \text{tensor field of rank } (p)\)

Exterior product of two forms \([\text{MTW} \Box 4.2]\)

Exterior derivative of a p-form \([\text{MTW} \S 4.1]\)

Ex 14.5, Ex 14.5
c) The Differential of a given function

[read p107-108 of the Apostol handout]

Then over a small \(|\Delta t| \ll 1\) segment of the curve \(c(t)\), the Taylor series expansion of \(f\) around \(p_0\) gets its dominant contribution from its linear part (a.k.a. "principal linear part")

\[
\begin{align*}
\Delta f &= f(p_0 + \Delta t) - f(p_0) \\
&= \frac{\partial f}{\partial x^i} \Delta x^i + \ldots \\
&= \langle \nabla f(p_0), \Delta t \rangle + \ldots
\end{align*}
\]

The integral curves of the vector field \(u\) satisfy

\[
\frac{dx^i}{dt} = u^i(p_0)
\]

Consequently

\[
\Delta f = \langle \nabla f(p_0), \Delta t \rangle + \ldots
\]

where \(u \in M_{p_0}^*\) = tangent space at \(p_0\)

\(=\) set of all vectors at \(p_0\)

\(
\{ \frac{\partial}{\partial x^i} \}\) basis for \(M_{p_0}^*\)

\(
\{ dx^i(p_0) \}\) dual basis for \(M_{p_0}^*\)

\[
\langle dx^i(p_0), \frac{\partial}{\partial x^j} \rangle = \frac{\partial x^i}{\partial x^j} = \delta^i_j
\]

Eq. (4.2) is a basis dependent expression.
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while Eq. (**) is basis independent.

This calculation leads to the following

Definition (Differential of \( f \) at a point)

The map

\[
\begin{align*}
\delta f &= \frac{\partial f}{\partial x^i} \bigg|_{p} \, dx^i \in T_p M^* \\
\text{a function of two variables}
\end{align*}
\]

\[\delta f : (p, u_0) \mapsto \delta f (p, u_0) = \langle df, u \rangle \]

\[
\begin{align*}
\frac{\partial f}{\partial x^i} &= \frac{\partial f}{\partial x^i} |_{p} \\
\text{is called the differential of \( f \) at \( p \).}
\end{align*}
\]

In descriptive terminology one has

\[
\frac{\partial f}{\partial x^i} \, dx^i = \text{rate of change of } f \text{ into an as-yet-unspecified direction at } p
\]

\[
u_{u_0} = u_{u_0} \frac{\partial}{\partial x^i} = \text{rate of change of an as-yet-unspecified quantity into the direction of } \{ u \} \text{ at } p
\]

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By allowing the point \( p \) to be any point of \( M \) one obtains the following

Definition (Differential 1-form)

A differential 1-form, say

\[
\omega (p) = \omega \mu (p) dx^\mu = \omega \mu (x^0, x^1, x^2, x^3) \, dx^\mu
\]

is the assignment of an element of \( T_p M^* \)

to each point \( p \) of the manifold.

Application to Line Integrals

Let \( c(t) = \{ c^0(t), \ldots, c^3(t) \} \) be a curve in \( M \) with tangent \( \frac{dc}{dt} = u : \{ u^0 = u^1 \} \).

Then \( A (p) \) arises from a line integral, i.e., an integral of a 1-form along the given curve as follows:

\[
\int_{c(t)} A (p) \, dx^\mu = \int_{c(t)} \omega \mu (c^0(t), \ldots, c^3(t)) \, \frac{dc^\mu}{dt} \, dt
\]

\[
= \int_{c(t)} \langle A, u \rangle \, dt
\]

Read MTW S4.1
Exterior Mathematics

The geometrical, i.e., coordinate-invariant formulation of multivariable integral calculus (line, surface, volume, and higher-dimensional integrals) requires the methods of exterior mathematics.

It consists of exterior algebra and exterior calculus.

Exterior Algebra

1. The Wedge Product (Read 4.1 in MTW)

A) Definition (Wedge Product)

Let $\alpha, \beta, \gamma \in V^*$

Then a two-vector (2-form) is

$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \in \Lambda^2 V^* \subset V^* \otimes V^*$

and b) a three-vector (3-form) is

$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) = \alpha \wedge \beta \wedge \gamma$

$= \alpha \otimes \beta \otimes \gamma + \beta \otimes \gamma \otimes \alpha + \gamma \otimes \alpha \otimes \beta$

$- \gamma \otimes \beta \otimes \alpha - \beta \otimes \alpha \otimes \gamma - \alpha \otimes \gamma \otimes \beta \in \Lambda^3 V^* \subset V^* \otimes V^* \otimes V^*$

are totally antisymmetric tensors of rank (2) and (3). They form tensor spaces, namely $\Lambda^2 V^*$ and $\Lambda^3 V^*$.

B) Similar definitions hold for wedge products of vectors.
Problem: Let \( \{ E_i \} \) basis for \( V \).
Let \( \{ \phi^i \} \) its dual basis for \( V^* \).

(1) Show: \( \{ \phi^1, \phi^2, \phi^3 \} \) is a basis for \( V^* \), the vector space of two forms.

(2) If \( \alpha \in \wedge^2 V \), find \( \alpha_{ij} \) such that
\[
\alpha = \frac{1}{2} \alpha_{ij} \wedge \phi^i \wedge \phi^j.
\]

Example: Integral of \( 2 \)-form over \( \Sigma \):

Define \( \Omega \)

\[
\int_{\Sigma} \omega = \int_{\Sigma} \frac{1}{2} \varepsilon_{ijk} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial w} \, du \, dv \, dw
\]

where \( \varepsilon_{ijk} \) is the antisymmetric Levi-Civita symbol.

MTW Box 4.4

\[
\text{MTW: } \text{Box 4.4, p. 278}
\]

\[
= \int_{\Sigma} \frac{1}{2} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial w} \, du \, dv \, dw
\]

\[
\text{MTW: } \text{Box 4.4, p. 278}
\]

\[
= \int_{\Sigma} \frac{1}{2} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial w} \, du \, dv \, dw
\]

\[
= \int_{\Sigma} \frac{1}{2} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial w} \, du \, dv \, dw
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\[
= \int_{\Sigma} \frac{1}{2} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} \frac{\partial x^k}{\partial w} \, du \, dv \, dw
\]

Notation: Modern math (e.g. Singer & Thorpe) Classical math (e.g. MTW) Diff' geometry in \( E^3 \) (e.g. Kaplan)