LECTURE 27

Differential 1-form as the linear approximation of a function at a point [MTW 9.4]

Integral of 1-form along a curve segment (i.e., "Line integral")

Differential 3-form, its integral over a 3-D domain [MTW P92-95]

[also MTW Box 4.4]
especially P117
Part C.

(a) The Differential of a given function
I read P107-108 of the Apostol handout

\[ C(t) \]

\[ \Delta x \]

\[ \Delta z \]

\[ x^1 \]

\[ x^2 \]

Given:

1) \( C: \mathbb{R} \rightarrow C(t) \)

2) \( \mathbb{R} \rightarrow [C^M(t)] \)

(b) Tangent at \( P_0 = C(t_0) \)

\[ x^M_0 = C_M(t_0) \]

\[ \frac{d}{d\tau} = \frac{dc^M}{d\tau} \bigg|_{t_0} = u^M(x_0) \frac{\partial}{\partial x^M} |_{P_0} \]

where \( u \in M_{P_0} = \text{tangent space at } P_0 \)

= set of all vectors at \( P_0 \)

\( \left\{ \frac{\partial}{\partial x^M} \right\} \) basis for \( M_{P_0} \)

\( \left\{ dx^M(P_0) \right\} \) dual basis for \( M_{P_0}^* \):

\[ \left\langle dx^M(P_0), \frac{\partial}{\partial x^N} \right\rangle = \frac{\partial x^M}{\partial x^N} |_{P_0} S^N \]
Then over a small ($|\Delta z| \ll 1$) segment of the curve $c(z)$ the Taylor series expansion of $f$ around $z_0$ gets its dominant contribution from its linear part (a.k.a. "principal linear part")

$$f(z) - f(z_0) = \frac{\partial f}{\partial x^m} \bigg|_{z_0} \Delta x^m + (\ldots) (\Delta z)^2 + \ldots$$

The integral curves of the vector field $u$ satisfy

$$\frac{dc^m}{dz} = u^m$$

Consequently,

$$f(z) - f(z_0) = \frac{\partial f}{\partial x^m} \bigg|_{z_0} \Delta x^m + (\ldots) (\Delta z)^2 + \ldots$$

$$\langle \frac{\partial f}{\partial x^m}, u^m \rangle \Delta z + (\ldots) (\Delta z)^2 + \ldots$$

$$\langle \delta f(z_0), u_{z_0} \rangle \Delta z + \text{etc}$$

Eq. (*) is a basis dependent expression.
while Eq. (**) is basis independent. This calculation leads to the following

**Definition (Differential of \( f \) at a point)**

The map

\[
\frac{df}{dx^m} \bigg|_{e_0} \in \mathcal{M}_{e_0}
\]

a function of two variables

\[
df: \mathcal{M} \times \mathcal{M}_{e_0} \to \mathbb{R}
\]

\((e_0, u_{e_0}) \mapsto df(e_0, u_{e_0}) = \left< \frac{df}{dx^m}, u \right|_{e_0}
\]

\[
= \left. \frac{df}{dx^m} u^m \right|_{e_0}
\]

is called the differential of \( f \) at \( e_0 \).

In descriptive terminology one has

\[
df \big|_{e_0} = \left. \frac{df}{dx^m} \right|_{e_0} \big|_{e_0} = \text{rate of change of } f \text{ into an as-yet-unspecified direction at } e_0
\]

\[
u \big|_{e_0} = u^m \bigg|_{e_0} \frac{\partial}{\partial x^m} = \text{rate of change of an as-yet-unspecified quantity into the direction of } \xi u \text{ at } e_0
\]
By allowing the point $p$ to be any point of $M$ one obtains the following.

**Definition (Differential 1-form)**

A differential 1-form, say

$$A(p) = A_\mu(p) \, dx^\mu = A_\mu(x^0, x^1, x^2, x^3) \, dx^\mu$$

is the assignment of an element of $M_0^*$ to each point $p$ of the manifold:

$$\mathcal{A} : M \times M_0 \to \mathbb{R} : (p, u_\mu) \mapsto \langle A_\mu, u_\mu \rangle = \langle A_\mu \, dx^\mu, u_\mu \rangle$$

**Application to Line Integrals.**

Let $c(t) = \{c^\mu(t)\}$ be a curve in $M$, with tangent $\frac{dc}{dt} = u : \{\frac{dc^\mu}{dt} = u^\mu\}$.

Then $A(p)$ arises from a line integral, i.e., an integral of a 1-form along the given curve, is defined as follows:

$$\int_{c(t)} A_\mu \, dx^\mu = \int_{t_1}^{t_2} A_\mu(c^0(t), \ldots, c^3(t)) \, \frac{dc^\mu}{dt} \, dt$$

$$= \int_{t_1}^{t_2} \left\langle A_\mu \, dx^\mu, \frac{dc^\mu}{dt} \frac{dc^\nu}{dt} \frac{\partial A_\mu}{\partial x^\nu} \right\rangle \, dt$$

$$= \int_{t_1}^{t_2} \left\langle A, u \right\rangle \, dt$$

Read MTW §4.1
Exterior Mathematics

The geometrical, i.e., coordinate invariant formulation of multivariable integral calculus (line, surface, volume, and higher dimensional integrals) requires the methods of exterior mathematics.

It consists of exterior algebra and exterior calculus.
The Wedge Product (Read 4.1 in MTW)

A) Definition (Wedge Product)

Let \( \alpha, \beta, \gamma \in V^* \)

Then a) a two-covector (2-form) is

\[
\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \quad E \Lambda^2 V^* \subset V^* \otimes V^*
\]

and b) a three-covector (3-form) is

\[
(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \equiv \alpha \wedge \beta \wedge \gamma
\]

\[
= \alpha \otimes \beta \otimes \gamma + \beta \otimes \gamma \otimes \alpha + \gamma \otimes \alpha \otimes \beta
\]

\[
- \gamma \otimes \beta \otimes \alpha - \beta \otimes \alpha \otimes \gamma - \alpha \otimes \gamma \otimes \beta \quad E \Lambda^3 V^* \subset V^* \otimes V^* \otimes V^*
\]

are totally antisymmetric tensors of

rank (2) and (3). They form tensor spaces, namely

\( \Lambda^2 V^* \) and \( \Lambda^3 V^* \).

B) Similar definitions hold for wedge products of vectors.
Example Integral of a 3-form over a volume

Let \( x^2(x_1^1, x_1^2, x_1^3) \) \( \ell = 1, 2, 3 \)

the coordinates of a typical point in the volume

\[ \text{Volume} = \{ x^2(x_1^1, x_1^2, x_1^3) : \ell = 1, 2, 3; \ x_0^1 < x_1^1 < x_1^1 \}
\]

and \( \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\} \) the coordinate induced basis which spans the vector space at each such typical point.

Definition (Volume integral)

\[
\int_{\text{Volume}} A(x^1, x^2, x^3) \, dx^1 \wedge dx^2 \wedge dx^3 = \iiint_{\text{Volume}} A(x^1) \left[ \frac{dx^1 \wedge dx^2 \wedge dx^3}{\partial x} + \text{even terms} \right] - \frac{dx^4 \wedge dx^1 \wedge dx^2}{\partial x} - \text{odd terms}
\]

\[
= \iiint_{\text{Volume}} \left( A(x^2(x_0^1, x_0^2, x_0^3)) \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\rangle \frac{dx^1 \wedge dx^2 \wedge dx^3}{\partial x^1} - \frac{dx^2 \wedge dx^1 \wedge dx^3}{\partial x^1} - \text{odd terms} \right) dx_0^1 dx_0^2 dx_0^3
\]

\[
\text{det} \begin{vmatrix}
\frac{\partial x^1}{\partial x_0^1} & \frac{\partial x^2}{\partial x_0^1} & \frac{\partial x^3}{\partial x_0^1} \\
\frac{\partial x^1}{\partial x_0^2} & \frac{\partial x^2}{\partial x_0^2} & \frac{\partial x^3}{\partial x_0^2} \\
\frac{\partial x^1}{\partial x_0^3} & \frac{\partial x^2}{\partial x_0^3} & \frac{\partial x^3}{\partial x_0^3}
\end{vmatrix}
\]

"Jacobian determinant"
This definition condenses into a single statement the following steps:

1. Substitute the parametrization

\[ x^2(\lambda_1, \lambda_2, \lambda_3) \]

into the form

\[ A = A(x^2) \frac{dx^1}{dx^3} \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} \]

\[ = A(x^2) \left[ \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} + d \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} d \frac{dx^3}{dx^3} \frac{dx^3}{dx^3} \right] \]

and collect terms

\[ A = A(x^2(\lambda_1, \lambda_2, \lambda_3)) \left[ \frac{dx^1}{dx^3} \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} \right] \]

\[ = A(x^2(\lambda_1, \lambda_2, \lambda_3)) \left[ \frac{dx^1}{dx^3} \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} \frac{dx^1}{dx^3} \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} \right] \]

\[ = A(x^2(\lambda_1, \lambda_2, \lambda_3)) \det \begin{vmatrix} \frac{dx^1}{dx^3} & \frac{dx^2}{dx^3} & \frac{dx^3}{dx^3} \\ \frac{dx^1}{dx^3} & \frac{dx^2}{dx^3} & \frac{dx^3}{dx^3} \\ \frac{dx^1}{dx^3} & \frac{dx^2}{dx^3} & \frac{dx^3}{dx^3} \end{vmatrix} \frac{dx^1}{dx^3} \frac{dx^2}{dx^3} \frac{dx^3}{dx^3} \]
2. Evaluate the integral over the 3-D domain in the usual way:

\[ \iiint_A (x^2(x^1, x^2, x^3)) \det \begin{vmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \end{vmatrix} \, dx^1 \, dx^2 \, dx^3 \]

Comment:
The above integral \( \iiint_A \) can be rewritten in terms of the evaluation of the rank (3) tensor \( A \) being evaluated on the trivector

\[ \frac{2}{2} - \frac{2}{2} - \frac{2}{2} \]

as follows

\[ A(x^1, x^2, x^3) \cdot \text{det above Jacobian matrix} \, dx^1 \, dx^2 \, dx^3 = \text{next page} \]

\[ \frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3} \]
evaluation of a tensor of rank $(3)$

\[ = \sum_{i} A(x^i, x^j, x^k) \quad \sum_{\alpha} \left( \frac{\partial}{\partial x^i} \right)^{\alpha} \left( \frac{\partial}{\partial x^j} \right)^{\beta} \left( \frac{\partial}{\partial x^k} \right)^{\gamma} \quad dx^i \otimes dx^j \otimes dx^k \quad \left( \frac{\partial}{\partial x^i} \right)^{\alpha} \left( \frac{\partial}{\partial x^j} \right)^{\beta} \left( \frac{\partial}{\partial x^k} \right)^{\gamma} \quad dx^i \otimes dx^j \otimes dx^k \]

in terms of

MTW's "restricted" sum:\n
\[ \sum_{ i < j < k } \sum_{ \ell \neq \alpha, \beta, \gamma } \left[ \frac{1}{3!} \varepsilon_{ijk} dx^\ell \wedge dx^i \wedge dx^j \wedge dx^k \right] \]

notation

evaluation of the rank $(3)$ tensor $A$ on the infinitesimal volume trivector

\[ dx^i \wedge dx^j \wedge dx^k = \varepsilon_{ijk} \quad \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \quad dx^i \Delta x^j \Delta x^k \]

\[ = A(x^i, x^j, x^k) \left( \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \right) \left( \frac{1}{3!} \varepsilon_{ijk} dx^\ell \wedge dx^i \wedge dx^j \wedge dx^k \right) \Delta x^i \Delta x^j \Delta x^k \]

\[ = A(x^i, x^j, x^k) \left( E_{\ell ij} \quad E^{\ell} \right) \left( \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \right) \left( \frac{2}{3!} \left( \frac{\partial}{\partial x^i} \right)^{\alpha} \left( \frac{\partial}{\partial x^j} \right)^{\beta} \left( \frac{\partial}{\partial x^k} \right)^{\gamma} \right) \Delta x^i \Delta x^j \Delta x^k \]

3-form

("egg-crate structure")

(see MTW P/17)
\[ \begin{align*}
= \int_0^L A \left( dx^1 dx^2 dx^3 \right) \partial \Delta^2 \partial \Delta^2 \partial \Delta^2 \\
= \int_0^L A \left( dx^1 dx^2 dx^3 \right) \frac{1}{2} \left( \partial^2 \partial^2 \partial^2 \right) \\
\end{align*} \]

\textbf{Comment:}

\( (a) \) in MTW define \( <, > \) on p.92-93

\( (m') \) a) \( dx^1 dx^2 dx^3 = \text{"elliptic structure"} \)

\( \text{MTW} \)

\( \text{Box 4.4} \)

\( b) \quad \frac{\partial}{\partial x^1} \land \frac{\partial}{\partial x^2} \land \frac{\partial}{\partial x^3} = \text{"oriented volume"} \)

spanned by

\[ \begin{align*}
\frac{\partial}{\partial x^1} &= \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^1} \\
\frac{\partial}{\partial x^2} &= \frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^2} \\
\frac{\partial}{\partial x^3} &= \frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^3}
\end{align*} \]

\text{Singer & Thorpe} \}

\text{MTW ch. 9}

\text{[various sources]}