Lecture 28 (continued)

EXTERIOR CALCULUS

1. Exterior product
2. Exterior derivative
I) Exterior Product

Given: anti-symmetric \( p \)-covector \( \omega_p \in \mathcal{P} \)  
\( q \)-covector \( \omega_q \in \mathcal{Q} \)

The "wedge" product gives rise to a \( p+q \)-covector \( \omega_{p+q} \) with the following defining properties:

(a) \( \omega_p \wedge \omega_q = (-1)^p q \omega_q \wedge \omega_p \)

(b) \( \omega_p \wedge (\omega_q + \omega_r) = \omega_p \wedge \omega_q + \omega_p \wedge \omega_r \)

(c) \( \omega \wedge (\omega \wedge \omega) = (\omega \wedge \omega) \wedge \omega \)

These three properties are a consequence of Lagrange's expansion of a determinant by complementary minors. Examples and explanation are found in "Lecture 28: Algebraic Supplement," including \( \text{det}(5 \times 5) = \sum \text{det}(2 \times 2) \cdot \text{det}(3 \times 3) \)
Examples of exterior product

1. \((A dx + B dy + C dz) \land (E dx + F dy + G dz)\)
   \[\implies (BG-CE) dy \land dz + (CE-AG) dz \land dx + (AF-BE) dx \land dy\]

2. \((A dx + B dy + C dz) \land (P dy \land dz + Q dz \land dx + R dx \land dy)\)
   \[\implies (AR + BQ + CP) dx \land dy \land dz\]

3. Let \(\sigma\) be a \(p\)-form of odd degree, \(p = \text{odd}\).
   \[p = 2n + 1\]

   Then \(\sigma \land \sigma = (-1)^p \sigma \land \sigma \implies \sigma \land \sigma = 0\), if \(p = \text{odd}\).

   But note that in general \(\sigma \land \sigma \neq 0\) if \(p = \text{even}\).

II. Exterior Derivatives

- Tensor field of totally antisymmetric \((p)\)-forms.

A. Let \(F^r(0) = \text{set of } p\text{-forms on } \mathcal{M}^n\)

The exterior differentiation operators:
\[d : F^r(0) \to F^{r+1}(0)\]

For any \(p\)-form, the one with the following defining properties:
4. Let 
\[ \omega^p = \sum_{i_1 < i_2 < \ldots < i_p} \frac{\delta_{i_1} \ldots \delta_{i_p}}{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} = p! \sum_{i_1 < i_2 < \ldots < i_p} \frac{\delta_{i_1} \ldots \delta_{i_p}}{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \]

Comment: MTW notation

\[ \sum_{i_1 < i_2 < \ldots < i_p} \text{"restricted sum"} \]

(2) \( \omega^q = \omega_{j_1 \ldots j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p} \)

Then
\[ \omega^p \wedge \omega^q = \sum_{i_1 = 1} \sum_{i_2 = 1} \ldots \frac{\delta_{i_1} \ldots \delta_{i_p}}{i_1 \ldots i_p} \omega_{j_1 \ldots j_q} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_q} \]
II Exterior derivative

\[ \text{Let } F^p(U) = \text{set of } p\text{-forms on } U \]

The exterior differential operator \( d \) is defined as follows:

\[ d: F^p(U) \rightarrow F^{p+1}(U) \]

Subject to the following mechanical rules:
The Four Rules of Exterior Differentiation

(i) \( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial z} \cdot dP = d \left( \frac{\partial P}{\partial x} \right) + \frac{\partial P}{\partial x} \cdot \frac{\partial P}{\partial y} \cdot \frac{\partial P}{\partial z} \)

(ii) \( d(x' + y) = dx' + dy \)

(iii) \( d(x' \cdot y) = dx' \cdot y + x' \cdot dy \)

(iv) For any \( \omega \), \( d(d\omega) = 0 \)

(v) For any function \( f \):

\[
\frac{\partial f}{\partial x} \cdot dx^2
\]

Note: A function is a tensor of rank zero; hence it is a zero-form (\( p = 0 \)).

Examples (using the above rules (i) - (v)):

(a) Let \( \omega = P(x, y, z) \cdot dx + Q(x, y, z) \cdot dy + R(x, y, z) \cdot dz \)

be a \( p=1 \) form.

Then

\[
d\omega = \left( \frac{\partial P}{\partial x} \cdot dx + \frac{\partial P}{\partial y} \cdot dy + \frac{\partial P}{\partial z} \cdot dz \right) \wedge dx
\]

\[
+ \left( \frac{\partial Q}{\partial x} \cdot dx + \frac{\partial Q}{\partial y} \cdot dy + \frac{\partial Q}{\partial z} \cdot dz \right) \wedge dy
\]

\[
+ \left( \frac{\partial R}{\partial x} \cdot dx + \frac{\partial R}{\partial y} \cdot dy + \frac{\partial R}{\partial z} \cdot dz \right) \wedge dz =
\]

\[
= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy
\]
b) Let \( \sigma = A dy \wedge dz + B dz \wedge dx + C dx \wedge dy \) be a \( p = 2 \) form.

Then,
\[
d\sigma = \left( \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz \right) \wedge dy \wedge dz
\]
\[
+ \left( \frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial y} dy + \frac{\partial B}{\partial z} dz \right) \wedge dz \wedge dx
\]
\[
+ \left( \frac{\partial C}{\partial x} dx + \frac{\partial C}{\partial y} dy + \frac{\partial C}{\partial z} dz \right) \wedge dx \wedge dy
\]
\[
= \left( \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} \right) dx \wedge dy \wedge dz \]