

Lecture 29

Parallel transport between tangent

spaces [MTW sects 8.3, 8.5, 10.3, 10.4; Box 10.2, 10.3]

Covariant differential of a ^{basis} vector [MTW § 14.5];

Covariant derivative [MTW sects 8.5,
10.3, 10.4]

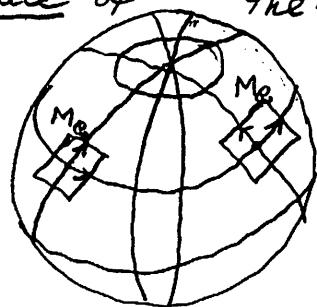
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Parallel transport (Read Ch. 10 in MTW)

Each point P of a manifold has associated with it a vector space, denoted by M_P .

This is the set of vectors tangent to their respective curves through P_0 .

Each of these vector spaces is called a tangent space of the manifold at the point P .



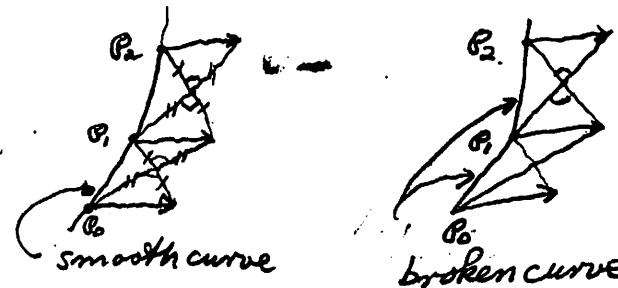
Although each point of a manifold has its own ^{natural *} vector space of tangent vectors, there is no isomorphism between different vector spaces.

The concept of parallelism, and hence the concept of parallel transport provides such an isomorphism. A parallel transport is also called a connection.

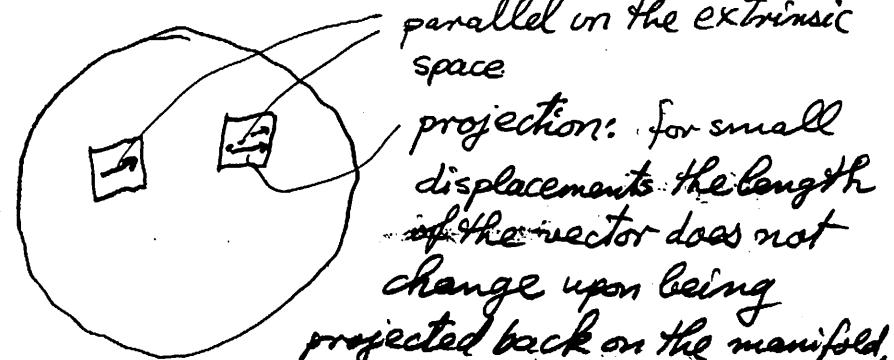
* natural = uniquely defined, non-arbitrary

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Example 1: Parallel transport via congruent geodesic triangles intrinsic to the manifold.



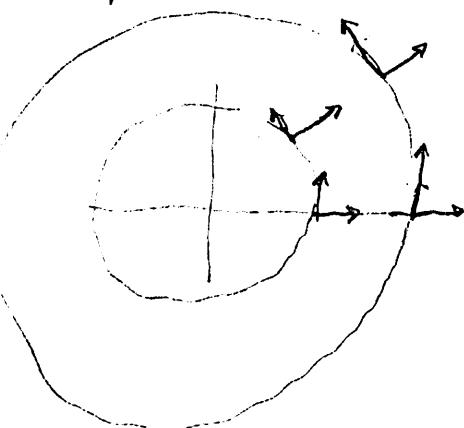
Example 2: Parallel transport based on translation and projection in a flat embedding space extrinsic to the manifold



II, Mathematical Formulation of - 32 -
Parallel Transport.

The concept of parallel transport arises in an arena consisting of a manifold M to each point p of which there is attached its tangent space M_p .

Having introduced in each of these vector spaces a basis continuously distributed over the whole manifold, let us consider two nearby vector space M_p and $M_{p+\delta p}$.



Polar coordinate induced field of basis vectors

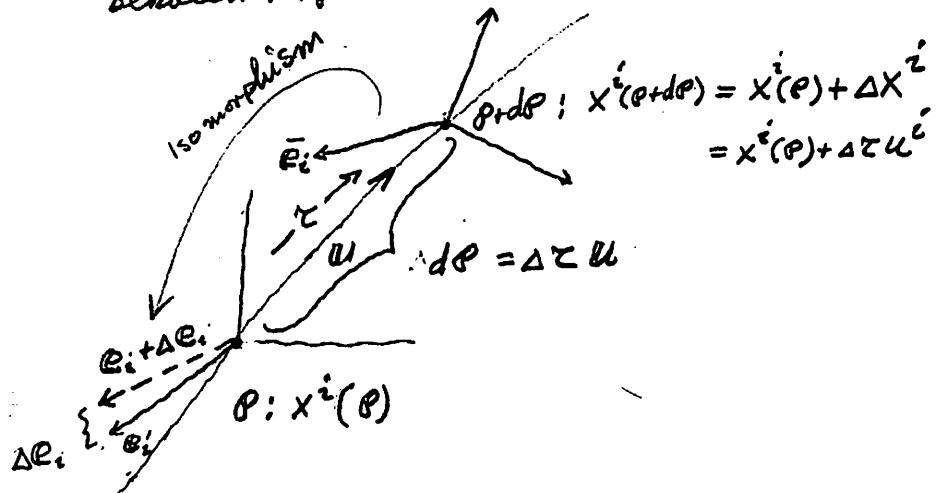
Mathematical Formulation of Parallel transport (continued)

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A parallel transport of vectors in M_θ to vectors in $M_{\theta+d\theta}$ is expressed by an isomorphism between M_θ and $M_{\theta+d\theta}$.

Let ρ and $\rho+d\rho$ be connected by an infinitesimal curve whose tangent is, say, u . Let $\{\mathbf{e}_i\}$ and $\{\bar{\mathbf{e}}_i\}$ be basis for M_θ and $M_{\theta+d\theta}$ respectively.

The parallel-transport-induced isomorphism between M_θ and $M_{\theta+d\theta}$ is as follows:



parallel-transport
induced
isomorphism:

$$M_\theta \xrightarrow{\text{isomorphism}} M_{\theta+d\theta}$$

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$$\bar{\mathbf{e}}_i \rightsquigarrow \mathbf{e}_i + \Delta \mathbf{e}_i = \mathbf{e}_i (\delta_i^j + \omega_i^j(\Delta))$$

$$(0, \dots, 0, 1, 0, \dots, 0) \rightsquigarrow (\omega_1^1(\Delta), \dots, \omega_{i-1}^{i-1}(\Delta), \omega_i^i(\Delta), \dots, \omega_n^n(\Delta))$$

i^{th} entry

↑
no sum!

Matrix representation of this isomorphism

$$\begin{bmatrix} \delta_i^j + \omega_i^j(\Delta) \end{bmatrix} = \begin{bmatrix} 1 + \omega_1^1(\Delta) & \omega_1^2(\Delta) & \dots & \omega_1^n(\Delta) \\ \omega_2^1(\Delta) & 1 + \omega_2^2(\Delta) & \dots & \omega_2^n(\Delta) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^1(\Delta) & \omega_n^2(\Delta) & \dots & 1 + \omega_n^n(\Delta) \end{bmatrix}$$

The coefficients $\omega_i^j(\Delta)$ determine $\Delta \mathbf{e}_i = \mathbf{e}_j \omega_i^j(\Delta)$. They depend
(i) on the chosen basis $\{\mathbf{e}_i\}$

(ii) linearly on the separation vector vector $dP = \Delta x u$: $\{\Delta x^k = \Delta x^k u^k\}$:
 $\omega_i^j(\Delta) = \Gamma_{jk}^i(\Delta) \Delta x^k + \dots + \Gamma_{jn}^i(\Delta) \Delta x^n =$
 $= \Gamma_{jk}^i(\Delta) \Delta x^k$, hence

(iii) linearly on the tangent vector

$$u = u^k \frac{\partial}{\partial x^k}$$

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Formulation of Parallel transport in terms
of (i) directional derivatives of the basis vectors
and (ii) differentials of the basis vectors.

GOTO ITEM 8 on p-6-

A) Reminder about the differential and the
directional derivative of a scalar.

(i) Directional derivative of $f(x^k)$ at ρ :

$$\nabla_u f = \lim_{\Delta \tau \rightarrow 0} \frac{f(\rho + \Delta \tau u) - f(\rho)}{\Delta \tau} = \lim_{\Delta \tau \rightarrow 0} \frac{\Delta f}{\Delta \tau}$$

$$= \lim_{\Delta \tau \rightarrow 0} \frac{f(x^k + \Delta \tau u^k) - f(x^k)}{\Delta \tau}$$

$$= \left. \frac{\partial f}{\partial x^k} \right|_{\rho} u^k$$

(ii) Differential of $f(x^k)$ at ρ :

$$\boxed{df = \frac{\partial f}{\partial x^k} dx^k} \quad (\text{B}, \text{E}) \Rightarrow df(x^k; u^l) = \\ = \langle df(\rho) | u^l \rangle \\ = \frac{\partial f}{\partial x^k} \langle dx^k | u^l \frac{\partial}{\partial x^l} \rangle \\ = \frac{\partial f}{\partial x^k} u^k.$$

is a real-valued function of 2 n
variables, namely $\{x^k\}$ and $\{u^l\}$,
with the dependence on $\{u^l\}$ linear.

Can be skipped.

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(iii) Relation between df and $\nabla_u f$:

$$\langle df | u \rangle = \frac{\partial f}{\partial x^k} u^k = \nabla_u f$$

Thus

$df = \nabla f$ = rate of change of f
into an as-yet-unspecified
direction.

B) Differential and the directional derivative
of a basis vector field

(i) Directional or covariant derivative
of i^{th} basis vector field. It is obtained
by taking infinitesimal differences
in the spirit of Newton, but only in
same vector space M_0 .

(Warning: subtracting vectors at different
points ρ & $\rho + d\rho$ is obviously not allowed)

$$\nabla_u e_i = \lim_{\Delta \tau \rightarrow 0} \frac{\Delta e_i}{\Delta \tau} = \lim_{\Delta \tau \rightarrow 0} \frac{e_i(\delta_i^j + \omega_i^j(\Delta \tau u)) - e_i}{\Delta \tau}$$

$$= e_j \cdot \frac{\omega_i^j(\Delta \tau u)}{\Delta \tau} = e_j \cdot \Gamma_{ik}^j \frac{\Delta x^k}{\Delta \tau}$$

$$= e_j \cdot \Gamma_{ik}^j u^k \text{ where } u^k \frac{\partial}{\partial x^k} = u$$

(ii) Compare this result with

$$\nabla_u f = \lim_{\Delta t \rightarrow 0} \frac{f(P + \Delta t \cdot u) - f(P)}{\Delta t} = \frac{df}{dt}$$

$$= \frac{\partial f}{\partial x^k} u^k \quad \text{where } u^k \frac{\partial}{\partial x^k} = u$$

which can be written as

$$\begin{aligned} \nabla_u f &= \left\langle \frac{\partial f}{\partial x^k} dx^k \mid u^k \frac{\partial}{\partial x^k} \right\rangle \\ &= \langle df \mid u \rangle \in \mathbb{R} \end{aligned}$$

in terms of the scalar valued differential form $df = \frac{\partial f}{\partial x^k} dx^k$.

We introduce vector-valued differential form

$$de_i = e_j \omega^j{}_i$$

in the same way, namely

$$\begin{aligned} e_j \Gamma^i{}_{ik} u^k &= e_j \left(\Gamma^i{}_{ik} dx^k \mid u^k \frac{\partial}{\partial x^k} \right) \\ &= e_j \left\langle \omega^i{}_k \mid u^k \frac{\partial}{\partial x^k} \right\rangle \end{aligned}$$

Thus we have

$$\nabla_u e_i = \langle de_i \mid u \rangle$$

where

$$\begin{aligned} de_i &= e_j \omega^j{}_i \\ &= e_j \Gamma^j{}_{ik} dx^k \end{aligned}$$

is the parallel transport induced by the vector valued differential 1-form

More precisely, we have

$$de_i = e_j \otimes \omega^j{}_i : M_g \rightarrow M_g$$

$$\Delta t u \wedge \langle de_i \mid \Delta t u \rangle = (\Delta e_i)$$

$$= \langle e_j \otimes \omega^j{}_i \mid \Delta t u \rangle$$

Given by transport on M .

$$= e_j \langle \omega^j{}_i \mid u \rangle \Delta t$$

$$= \nabla_u e_i \Delta t$$

$\therefore \nabla_u e_i = \text{rate of change of } e_i \text{ due to parallel transport into direction } u.$

$\Gamma^j{}_{ik} dx^k$ is called the connection 1-form which is a mathematical expression of the law of parallel transport.