Parallel transport between tangent spaces \([\text{MTW sects. 8.3, 8.5, 10.3, 10.4; Box 10.2, 10.3}]\)

covariant differential of a vector \([\text{MTW sect 14.5}]\)

covariant derivative \([\text{MTW sect 8.5, 10.3, 10.4}]\)
Parallel transport (Read Ch. 10 in MTW)

Each point of a manifold has an associated vector space, denoted by $M_p$. This is the set of vectors tangent to their respective curves through $p_0$. Each of these vector spaces is called a tangent space of the manifold at the point $p$. Although each point of a manifold has its own vector space of tangent vectors, there is no isomorphism between different vector spaces. The concept of parallelism, and hence the concept of parallel transport, provides such an isomorphism. A parallel transport is also called a connection.

* natural = uniquely defined, non-arbitrary

Example 1: Parallel transport via congruent geodesic triangles intrinsic to the manifold.

Example 2: Parallel transport based on translation and projection in a flat embedding space extrinsic to the manifold.
II. Mathematical Formulation of Parallel Transport

The concept of parallel transport arises in an arena consisting of a manifold M to each point P of which there is attached its tangent space M₀.

Having introduced in each of these vector spaces a basis continuously distributed over the whole manifold, let us consider two nearby vector spaces M₀ and M₀ + dP.

Polar coordinate induced field of basis vectors
Mathematical Formulation of Parallel transport

A parallel transport of vectors in $M_0$ to vectors in $M_0 + \delta \sigma$ is expressed by an isomorphism between $M_0$ and $M_0 + \delta \sigma$.

Let $p$ and $p + \delta \sigma$ be connected by an infinitesimal curve whose tangent is $\delta z / \delta \sigma$.

Let $\{e_i\}$ and $\{e_i'\}$ be bases for $M_0$ and $M_0 + \delta \sigma$ respectively.

The parallel-transport-induced isomorphism between $M_0$ and $M_0 + \delta \sigma$ is as follows:

Parallel-transport induced isomorphism:

$$M_0 + \delta \sigma \rightarrow M_0$$

$$e_i \rightarrow e_i + \delta e_i = e_i (\delta z^k / \delta \sigma + \omega_i^k (\sigma))$$

$$\begin{pmatrix}
\omega_1^k (\sigma) \\
\omega_2^k (\sigma) \\
\vdots \\
\omega_n^k (\sigma)
\end{pmatrix} \rightarrow \begin{pmatrix}
\omega_1^k (\sigma) \\
\omega_2^k (\sigma) \\
\vdots \\
\omega_n^k (\sigma)
\end{pmatrix}$$

Matrix representation of this isomorphism:

$$\begin{bmatrix}
\delta z^1 / \delta \sigma + \omega_1^1 (\sigma) \\
\delta z^2 / \delta \sigma + \omega_1^2 (\sigma) \\
\vdots \\
\delta z^n / \delta \sigma + \omega_1^n (\sigma)
\end{bmatrix} = \begin{bmatrix}
\omega_1^1 (\sigma) & 1 & \omega_2^1 (\sigma) & \cdots & \omega_1^n (\sigma) \\
\omega_1^2 (\sigma) & 1 & \omega_2^2 (\sigma) & \cdots & \omega_2^n (\sigma) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_1^n (\sigma) & 1 & \omega_2^n (\sigma) & \cdots & 1 + \omega_n^n (\sigma)
\end{bmatrix}$$

The coefficients $\omega_i^k (\sigma)$ determine $\delta e_i = e_i' - \omega_i^k (\sigma) e_k$.

They depend:

(i) on the chosen basis $\{e_i\}$

(ii) linearly on the separation vector $\delta \sigma = \delta z^k (\sigma) e_k$.

$$\omega_i^k (\sigma) = \frac{\partial}{\partial z^k} (\delta z^i / \delta \sigma) = \frac{\partial}{\partial z^k} (\delta z^i / \delta \sigma) = \frac{\partial}{\partial z^k} (\delta z^i / \delta \sigma)$$

(iii) linearly on the tangent vector $u = u_k \frac{\partial}{\partial x^k}$. 
Formulation of Parallel transport in terms of (i) directional derivatives of the basis vectors and (ii) differentials of the basis vectors.

A) Reminder about the differential and the directional derivative of a scalar:

(i) Directional derivative of \( f(x^k) \) at \( \bar{\phi} \):

\[
\nabla_{u^k} f = \lim_{\Delta x^k \to 0} \frac{f(\bar{\phi} + \Delta x^k) - f(\bar{\phi})}{\Delta x^k} = \lim_{\Delta x^k \to 0} \frac{\Delta f}{\Delta x^k} = \left. \frac{\partial f}{\partial x^k} \right|_{\bar{\phi}} u^k
\]

(ii) Differential of \( f(x^k) \) at \( \bar{\phi} \):

\[
\delta f = \left. \frac{\partial f}{\partial x^k} \right|_{\bar{\phi}} \delta x^k
\]

is a real-valued function of \( 2m \) variables, namely \( \{x^k\} \) and \( \{u^k\} \) with the dependence on \( \{u^k\} \) linear.

B) Differential and the directional derivative of a basis vector field

(iii) Relation between \( df \) and \( \nabla_{u^k} f \):

\[
\left< df(\bar{\phi}) \right> = \frac{\partial f}{\partial x^k} u^k = \nabla_{u^k} f
\]

Thus

\[
df = \nabla f = \text{rate of change of } f \text{ into an as-yet-unspecified direction.}
\]

B) Differential and the directional derivative of a basis vector field

(iv) Directional or covariant derivative of \( z^k \) basis vector field, it is obtained by taking infinitesimal differences in the spirit of Newton, but only in same vector space \( \mathcal{M} \).

(Warning: subtracting vectors at different points \( \bar{\phi} \) + \( d\bar{\phi} \) is obviously not allowed)

\[
\nabla_{u^k} (e_j) = \lim_{\Delta x^k \to 0} \frac{\Delta e_j}{\Delta x^k} = \lim_{\Delta x^k \to 0} \frac{e_j (\delta^i_j + \omega^i_j(\Delta x^k)) - e_i}{\Delta x^k}
\]

\[
= e_j \left. \frac{\partial \omega^i_j}{\partial x^k} \right|_{\bar{\phi}} = e_j \Gamma^i_{jk} \frac{\Delta x^k}{\Delta x^l}
\]

\[
= e_j \Gamma^i_{jk} u^k \text{ where } u^i = \left. \frac{\partial \bar{\phi}}{\partial x^i} \right|_{\bar{\phi}}
\]
We introduce vector-valued differential form in terms of the scalar valued differential which can be written as

\[ \frac{df}{dx} = \text{constant} \]

where the partial derivative is

\[ \frac{df}{dx} = \frac{\partial f}{\partial x} \]

hence, we have

\[ \frac{df}{dx} = \text{constant} \]

Then we have

\[ \frac{df}{dx} = \text{constant} \]

More precisely, we have

\[ \frac{df}{dx} = \text{constant} \]

The parallel transport induced is the parallel transport differential 1-form.

\[ \omega^1 = \omega^1 \text{dx} \]

where

\[ \omega^1 = \text{constant} \text{dx} \]

and \( \omega^2 = \text{constant} \text{dx} \) is called the connection 1-form which is a mathematical expression of the law of parallel transport.