

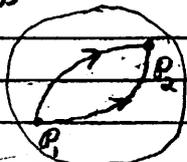
Lecture 30

- A) Parallelism [MTW sects 8.3, 8.5, 10.2, 10.4, Box 10.2, 10.3, Fig 10.2]
- B) The covariant differential of a vector field [MTW sect. 14.5]
- C) The covariant derivative [MTW § 8.5, 10.3, 10.4]
[In MTW Read Chapter 10]

A) PARALLELISM

30.1a

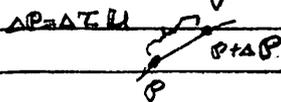
Parallelism is a concept which establishes a one-to-one relation between vectors in tangent spaces at different points of a manifold, points which can be connected by curves between them. These curves are in general different



and thus give rise to different isomorphisms between the tangent spaces.

However, for infinitesimal close points

the connecting curves are unique,



are determined: its tangent vector and are given by the relation linear in the curve parameter s

$$\Delta P = \Delta T U; \{ \Delta X^R = \Delta T U^R \}$$

30.1b

INSERT a footnote (for page 30.1a)

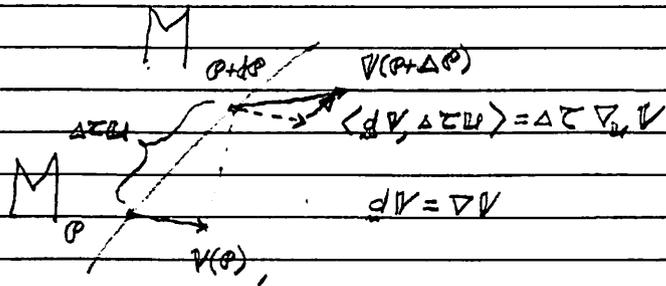
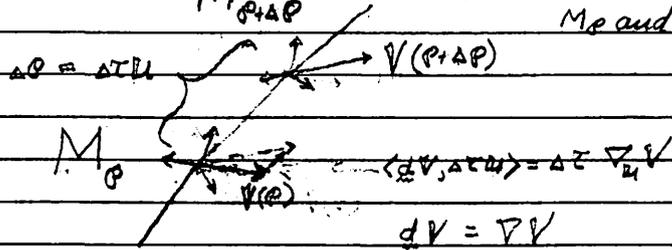
\begin { footnote }

It is understood but is worth mentioning that the concept "parallelism" is applicable to vector spaces other than tangent spaces attached to the points of a manifold. Aside from differentiability restrictions the only restriction is that these vector spaces be copies of one and the same vector space, even if it is a complex vector space, e.g. the wave mechanical state space of a particle at each point.

\end { footnote }

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Figure 1: A T -parametrized family of moving frames $M_{P+\Delta P}$ and two of its vector spaces, M_P and $M_{P+\Delta P}$



Parallelism along a given curve gives rise to a law of parallel transport of vectors along the curve. Vectors in different vector spaces, say M_P and $M_{P+\Delta P}$ cannot be subtracted. However, a vector and a parallel transported image

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from a neighboring vector space can.

This is because ^{the} parallel transport along the curve places the parallel image from that neighboring space into the same vector space, there the difference between

(i) a preexisting vector and (ii) ^{the} its image due to parallel transport law is a mere subtraction in the same vector space.

This subtraction is depicted in the two figures on page 30.2. The difference equals the amount by which a vector is not parallel to its neighbor.

The mathematically most direct way of expressing this subtraction process is by exhibiting the differential of a vector field, say

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$$V = e_i v^i$$

As was shown in Lecture 29 parallelism between the vectors in two vector space is expressed in terms of an isomorphism between the two vector spaces

$$e_i + \langle de_i, \Delta \tau U \rangle \rightsquigarrow \bar{e}_i \quad i=1, \dots, n$$

where $C \in M_p$ $\bar{C} \in M_{q,p}$

where

$$e_i + \langle de_i, \Delta \tau U \rangle = e_i \left(\delta_i^j + \langle \omega_{mi}^j, \Delta \tau U \rangle \right) \quad (*)$$

The matrix of this isomorphism

$$\left[\delta_i^j + \langle \omega_{mi}^j, \Delta \tau U \rangle \right],$$

differs from the identity matrix $[\delta_i^j]$

by

$$\left[\langle \omega_{mi}^j, \Delta \tau U \rangle \right]$$

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Each of its entries is a linear function,
the connection 1-form of covectors

$$\omega_{im}^j: \quad \{i, j\} = 1, \dots, n$$

evaluated on the displacement vector

$$\Delta P = \Delta \tau U,$$

which separates the neighboring
tangent spaces M_p and $M_{p+\Delta P}$.

Relative to a coordinate basis $\{e_i := \frac{\partial}{\partial x^i}\}$

and its dual basis $\{\omega_{im}^j = dx^j\}$ a

connection 1-form of covectors assumes
the form

$$\omega_{im}^j = \Gamma_{ik}^j(x) dx^k.$$

Here, the coefficients Γ_{ik}^j are called
the Christoffel symbols. They characterize
the parallelism between any pair of

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adjacent tangent spaces, such as M_p
and $M_{p+\Delta P}$ in Figure 1 on p 30.2,
(on page 30.4)

The requirement that Eq. (*) holds for
any displacement $\Delta P = \Delta \tau U$ implies that

- (1) $de_i = e_j \omega_{im}^j$ (in terms of the connection 1-form of covectors)
- (2) $de_i = e_j \hat{\Gamma}_{ik}^j \omega_{im}^k$ (relative to a generic basis)
- (3) $de_i = e_j \Gamma_{ik}^j dx^k$ (relative to a coordinate basis)

The mathematization of parallel transport
is achieved by any ^{one} of the three equations (1)-(3).

This result is expressed by the following

Proposition (Parallelism):

The parallelism between adjacent tangent
spaces is determined if and only if

one knows the connection 1-forms

$\omega^a{}_b$ in Eq. (1), or the connection coefficients

$\Gamma^a{}_{bc}(x)$ (which are functions of position),

in Eq. (2), or the Christoffel symbols

$\Gamma^a{}_{bc}(x)$ (which are functions of position)

in Eq. (3).

B) The Covariant Differential of a Vector Field.

A vector field, say

$$V = \mathcal{E}_i v^i(x), \quad (**)$$

is the assignment of a tangent vector V_p to each tangent space M_p of the

manifold. One now asks the question:

Are the vectors assigned by V to adjacent tangent spaces, say M_p and $M_{p+\delta p}$,

parallel to each other?

This question is answered by calculating the differential dV of V in Eq. (**) above.

This expression is a sum of products and the calculation parallels the ^{familiar} differentiation of the product of scalars.

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The differential de_i of e_i is known and given by

$$de_i = e_j \omega^j_i = e_j \Gamma^j_{ik} dx^k \quad i=1, \dots, n$$

This is the law of parallel transport, (assumed to be)

known and given. Consequently,

the "covariant" differential of V is

$$dV = d(e_i v^i)$$

$$= e_i dv^i + de_i v^i$$

$$= e_i \frac{\partial v^i}{\partial x^k} dx^k + e_j \omega^j_i v^i$$

$$= \quad \quad + e_j \Gamma^j_{ik} dx^k v^i \quad i \leftrightarrow j$$

$$= e_i \left(\frac{\partial v^i}{\partial x^k} + \Gamma^i_{jk} v^j \right) dx^k$$

This is a vector-valued differential form.

The coefficients

$$\frac{\partial v^i}{\partial x^k} + \Gamma^i_{jk} v^j \equiv v^i_{;k}$$

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are called the components of the covariant derivative of V . We

have

$$dV = e_i v^i_{;k} dx^k$$

(that there are)

(namely,

Note two contributions, (a) $\frac{\partial v^i}{\partial x^k} dx^k$ which is

due to the change in the components v^i of

V , and (b)

$$\Gamma^i_{jk} v^j dx^k$$

(the change)

which is due to the nonparallelism (rotation,

rescaling) of the basis elements $\{e_i\}$.

The vector differential dV expresses

the deviation away from parallelism of the vector field V . More precisely,

$dV =$ the change of V away from being parallel, due to the displacement into an as-yet-unspecified direction.

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The vector differential dV expresses
the vectorial change away from parallelism

of the vector field V . More precisely
 $dV: M \times M_p \rightarrow M_p; (p, u) \mapsto \langle dV, u \rangle_p = e_i v^i \cdot \frac{\partial}{\partial x^k} u^k$
is the means for calculating that

change once an infinitesimal displacement, say

$$\Delta p = \Delta \tau u: \{\Delta x^k = \Delta \tau u^k; k=1, \dots, n\},$$

is given. That calculated change is

designated by

$$\langle dV, \Delta \tau u \rangle = \text{"vectorial change away
from parallelism of the
vector field } V, \text{ the change
due to the displacement
}\Delta \tau u\text{"}$$

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and according to the expression on P 30.4
is given by

$$\begin{aligned} \langle dV, \Delta \tau u \rangle &= \langle e_i v^i;_k dx^k, \Delta \tau u^l \frac{\partial}{\partial x^l} \rangle \\ &= e_i v^i;_k u^l \langle dx^k, \frac{\partial}{\partial x^l} \rangle \Delta \tau \end{aligned}$$

Taking advantage of the differential
duality $\langle dx^k, \frac{\partial}{\partial x^l} \rangle = \delta^k_l$

one obtains the vectorial change

$$\langle dV, \Delta \tau u \rangle = e_i v^i;_k u^k \Delta \tau$$

as one moves from p to $p + \Delta p$ by the
amount $u \Delta \tau$ as depicted by either
of the figures on P 30.2.

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Note that $de_i = e_j \omega_j^i = e_j \Gamma_{iR}^j dx^R$,

$$dV = e_j v_j^i dx^i,$$

unlike

$$\sigma = \sigma_j \omega_j^i = \sigma_j \Gamma_{iR}^j dx^R$$

$$df = \frac{\partial f}{\partial x^k} dx^k,$$

are vector-valued differential forms,
not scalar-valued diff'l forms. They
are obviously vector-valued

because their values on a vector, say

$$u = u^k \frac{\partial}{\partial x^k} \text{ are}$$

$$\langle de_i, u \rangle = \langle e_j \Gamma_{iR}^j dx^R, u \rangle$$

$$= e_j \Gamma_{iR}^j u^R$$

$$\langle dV, u \rangle = \langle e_j v_j^i dx^i, u \rangle$$

$$= e_j v_j^i u^i$$

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If u refers to a direction of motion,
the the vectorial value of dV refers
to

$\langle dV, u \rangle =$ vectorial rate of change
of V away from being parallel
due to motion with velocity u .

If $\Delta P = \Delta \tau u$, $\{\Delta x^k = \Delta \tau u^k\}$

is the displacement from P to $P + \Delta \tau$,

then the amount of vectorial change in

V is simply

$$\langle dV, \Delta \tau u \rangle = \langle dV, u \rangle \Delta \tau.$$

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c) The covariant derivative operator
as a
Directional Derivative.

Recall the directional derivative
of a scalar function f and its relation

to the differential $df = \frac{\partial f}{\partial x^k} dx^k$;

$$\langle df, u \rangle = \left\langle \frac{\partial f}{\partial x^k} dx^k, u^l \frac{\partial}{\partial x^l} \right\rangle = \frac{\partial f}{\partial x^k} u^l \frac{\partial x^k}{\partial x^l}$$

$$= u^l \frac{\partial f}{\partial x^l}$$

$$\langle df, u \rangle \equiv \nabla_u f \quad (\equiv u(f) \text{ in the notation on P.26.5})$$

The covariant derivative extends the
~~familiar~~
directional derivative from scalars to
vectors. One introduces from P.30.13

$$\langle dV, u \rangle \equiv \nabla_u V$$

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Problem (Components of ∇)
Compute $\nabla_{e_i} e_j$.

Solution:

$$\nabla_{e_R} e_i = \langle de_i, e_R \rangle$$

$$= \left\langle e_j \Gamma_{i2}^j dx^2, \frac{\partial}{\partial x^R} \right\rangle$$

$$= e_j \Gamma_{i2}^j \left\langle dx^2, \frac{\partial}{\partial x^R} \right\rangle$$

$$= e_j \Gamma_{i2}^j$$

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Like the differential operator d ,
 the direction derivative operator pertains
 to ^{both} scalar fields and vector fields. It
 therefore is just as powerful. Its 4 defining
 properties are given by the following

Definition ($\nabla_u V$)

- (1) $\nabla_u (V_1 + V_2) = \nabla_u V_1 + \nabla_u V_2$ (distributivity)
- (2) $\nabla_{u_1 + u_2} V = \nabla_{u_1} V + \nabla_{u_2} V$ \checkmark
- (3) For any scalar field f
 $\nabla_{fu} V = f \nabla_u V$ (pointwise linearity)
- (4) $\nabla_u (fV) = \nabla_u(f)V + f \nabla_u V$ (product ("Leibnitz")
 rule)

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Problem (components of ∇)

Given: $u = e_k u^k$
 $v = e_i v^i$

Exhibit $\nabla_u v$

Solution:

$$\nabla_u v = \nabla_{e_k u^k} v$$

$$= \nabla_{e_k} (e_i v^i) u^k$$

$$= [e_i \nabla_{e_k} v^i + \nabla_{e_k} (e_i) v^i] u^k$$

$$= [e_i \frac{\partial v^i}{\partial x^k} + e_j \Gamma_{ik}^j v^i] u^k$$

$$\nabla_u v = e_i \left[\frac{\partial v^i}{\partial x^k} + \Gamma_{ik}^j v^i \right] u^k$$

$$\nabla_u v = e_i \cdot v^i_{;k} u^k$$

Here $v^i_{;k}$ are the components of ∇ .