Lecture 30

A) Parallelism [MTW sects 8.3, 8.5, 10.2, 10.4, Box 10.2, 103, Fig 10.2]

B) The covariant differential of a vector field [MTW sect. 14.5]

C) The covariant derivative [MTW § 8.5]

[In MTW Read Chapter 10]
A) PARALLELISM

Parallelism is a concept which establishes a one-to-one relation between vectors in tangent spaces at different points of a manifold, points which can be connected by curves between them. These curves are in general different, and thus give rise to different isomorphisms between the tangent spaces. However, for infinitesimal close points, the connecting curves are unique, and determined by its tangent vector and are given by the relation linear in the curve parameter: 

\[ \Delta \mathbf{p} = \Delta \mathbf{u} \cdot \mathbf{X} = \Delta \mathbf{u} \cdot \mathbf{X} \]

INSERT a footnote (for page 3a/16)

begin footnote:

It is understood but a worth mentioning that the concept "parallelism" is applicable to vector spaces other than tangent spaces attached to the points of a manifold. Aside from differentiability restrictions the only restriction is that these vector spaces be copies of one and the same vector space, even if it is a complex vector space, e.g. the wave mechanical state space of a particle at each point.

end footnote.
Figure 1: A $t$-parametrized family of moving frames and two of its vector spaces, $M_{\phi}$ and $M_{\phi + \Delta \phi}$.

Parallelism along a given curve gives rise to a law of parallel transport of vectors along the curve. Vectors in different vector spaces, say $M_{\phi}$ and $M_{\phi + \Delta \phi}$, cannot be subtracted. However, a vector and a parallel transported image
from a neighboring vector space can.

As was shown in Lecture 28 parallelism,

This is because parallel transport along

between the vectors in two vector space

the curve places the parallel image from

is expressed in term of an isomorphism between

that neighboring space into the same

the two vector spaces

vector space, there the difference between

(i) a preexisting vector and (ii) its image

the due to parallel transport law is a mere

where

(ii) due to parallel transport law is a mere

subtraction in the same vector space.

subtraction in the same vector space.

The subtraction is depicted in the two

The difference equals the amount by which a

figures on page 30.2. A vector is not parallel

to its neighbor.

The mathematically most direct way

The matrix of this isomorphism is

of expressing this subtraction process

is by exhibiting the differential

differs from the identity matrix \([S^2]\)

of a vector field, say

by

\[\begin{bmatrix} S^2 + \langle \omega \cdot \Delta U \rangle \\ \langle \omega \cdot \Delta U \rangle \end{bmatrix}\]
Each of its entries is a linear function, the connection 1-form of covectors

\[ \omega^i : \delta_j = 1, \ldots, n \]

evaluated on the displacement vector

\[ \Delta \delta = \Delta z \cdot \delta_i \]

which separates the neighboring tangent spaces \( M_p \) and \( M_{p + \Delta \delta} \).

Relative to a coordinate basis \( \delta_i : = \frac{\partial}{\partial x^i} \)

and its dual basis \( \omega^i = dx^i \cdot \delta_i \) a connection 1-form of covectors assumes the form

\[ \omega^i : = \Gamma^i_{jk} (x) \; dx^j \]

Here, the coefficients \( \Gamma^i_{jk} \) are called the Christoffel symbols. They characterize the parallelism between any pair of adjacent tangent spaces.

\[ \Delta p = \Delta z \cdot \delta_i \]

The requirement that Eq. (4) holds for any displacement \( \Delta p = \Delta z \cdot \delta_i \) implies that

\( (1) \) \hspace{1cm} \( \delta e_i = e_j \cdot \omega^j \cdot \delta_i \) (in terms of the connection 1-form of the covectors)

\( (2) \) \hspace{1cm} \( \delta e_i = e_j \cdot \Gamma^j_{ik} (x) \cdot \delta_i \) (relative to a generic basis)

\( (3) \) \hspace{1cm} \( \delta e_i = e_j \cdot \Gamma^j_{ik} \cdot dx^k \) (relative to a coordinate basis)

The mathematicalization of parallel transport is achieved by any of the three equations (1-3).

This result is expressed by the following Proposition (Parallelism):

The parallelism between adjacent tangent spaces is determined if and only if...
one knows the connection 1-forms
\[ \omega^{a}_{\ bc}(x) \]
in Eq. (1), or the connection coefficients
\[ \Gamma^{a}_{\ bc}(x) \]
(\text{which are functions of position})
in Eq. (2), or the Christoffel symbols
\[ \Gamma^{a}_{\ bc}(x) \]
(\text{which are functions of position})
in Eq. (3).

3.0.7

B) The Covariant Differential of a Vector Field.

\[ \mathbf{V} = \mathbf{e}_{i} \mathbf{v}^{i} \]
is the assignment of a tangent vector \( v \) to each tangent space \( M_{p} \) of the manifold. One now asks the question: Are the vectors assigned by \( V \) to adjacent tangent spaces, say \( M_{p} \) and \( M_{p+1} \), parallel to each other?

This question is answered by calculating the differential \( dV \) of \( V \) in Eq. (**) above.

This expression is a sum of products and familiar. The calculation parallels the differentiation of the product of scalars.
The differential $d\mathbf{e}_i$ of $\mathbf{e}_i$ is known and given by

$$d\mathbf{e}_i = e_j \omega^i_j = e_j \Gamma^i_{jk} \, dx^k \quad i, j, k = 1, \ldots, n.$$ 

This is the law of parallel transport, known and given. Consequently, the covariant differential of $V$ is

$$dV = d(\mathbf{e}_i \cdot V^i)$$

$$= e_i \, dV^i + d\mathbf{e}_i \cdot V^i$$

$$= e_i \left( \frac{\partial V^i}{\partial x^j} \, dx^j + e_j \omega^i_j \, V^k \right) \quad i, j, k = 1, \ldots, n$$

$$= e_i \left( \frac{\partial V^i}{\partial x^j} \, dx^j + \Gamma^i_{jk} \, V^k \right) dx^i$$

This is a vector-valued differential form. The coefficients

$$\frac{\partial V^i}{\partial x^j} + \frac{\partial V^j}{\partial x^i} - \frac{\partial V^k}{\partial x^i} \frac{\partial V^i}{\partial x^k} = 0$$

are called the components of the covariant derivative of $V$. We have

$$dV = \overline{\mathbf{e}_i} \omega^i_j \, dx^j \quad \text{namely}$$

Note two contributions, $\frac{\partial V^i}{\partial x^j} \, dx^j$, which is due to the change in the components $V^i$ of $V$, and (b) $\Gamma^i_{jk} \, V^k \, dx^i$, which is due to the nonparallelism (rotation, rescaling) of the basis elements $\{\mathbf{e}_i\}$. The vector differential $dV$ expresses the deviation away from parallelism of the vector field $V$. More precisely,

$$dV = \text{the change of } V \text{ away from being parallel due to the displacement into an as-yet-unspecified direction.}$$
The vector differential $dV$ expresses the vectorial change away from parallelism of the vector field $V$. More precisely, the means for calculating that change once an infinitesimal displacement, say $\delta x = \delta x^k \Delta x_k$, is given by

$$
<\Delta V, \Delta x> = \sum_{k=1}^{\infty} \varepsilon_i \frac{\partial v^i}{\partial x^k} \Delta u^k \Delta x^k
$$

and according to the expression on p. 30.4, is given by

$$
<\Delta V, \Delta x> = \sum_{k=1}^{\infty} \varepsilon_i \frac{\partial v^i}{\partial x^k} \Delta u^k \Delta x^k
$$

Taking advantage of the differential duality $\langle dx^k, \frac{\partial}{\partial x^k} \rangle = \delta^i_e$, one obtains the vectorial change

$$
<\Delta V, \Delta x> = \sum_{k=1}^{\infty} \varepsilon_i \frac{\partial v^i}{\partial x^k} \Delta u^k \Delta x^k
$$

as one moves from $P$ to $P + \Delta x$ by the amount $\Delta x^k$ as depicted by either of the figures on p. 30.2.
Note that
\[ \text{d}v = e^i \cdot \text{d}x^i = e^i \cdot \Gamma^j_{ik} \text{d}x^k \]

Unlike
\[ \sigma = \sigma^j \cdot \text{d}x^j = e^j \cdot h^k \text{d}x^k \]
\[ \text{d}f = \frac{\partial f}{\partial x^i} \text{d}x^k \]

are vector-valued differential forms.

not scalar-valued differential forms they are obviously vector-valued because their values on a vector say
\[ u = u^i \frac{\partial}{\partial x^i} \text{are} \]
\[ \langle \text{d}e^i, u \rangle = \langle e^j \cdot \Gamma^i_{jk} \text{d}x^k, u \rangle = e^j \cdot \Gamma^i_{jk} \cdot u^k \]

\[ \langle \text{d}v, u \rangle = \langle e^j \cdot \text{d}x^j, u \rangle \]
\[ = e^j \cdot \text{d}x^j \cdot u^k \]

If \( u \) refers to a direction of motion the the vectorial value of \( \text{d}v \) refers to
\[ \langle \text{d}v, u \rangle = \text{vectorial rate of change} \]
\[ \text{of } v \text{ away from being parallel due to motion with velocity } u. \]

If
\[ \Delta \mathbf{r} = \Delta x^1 \mathbf{e}_1 + \Delta x^2 \mathbf{e}_2 + \Delta x^3 \mathbf{e}_3 \]

is the displacement from \( P \) to \( P + \Delta \mathbf{r} \),

then the amount of vectorial change in \( v \) is simply
\[ \langle \text{d}v, \Delta \mathbf{r} u \rangle = \langle \text{d}v, u \rangle \Delta \mathbf{r}. \]
0) The covariant derivative operator as a directional derivative.

Recall the directional derivative of a scalar function $f$ and its relation to the differential $df = \frac{\partial f}{\partial x^k} dx^k$.

\[
\langle df, u \rangle = \left\langle \frac{\partial f}{\partial x^k}, u^k \frac{\partial}{\partial x^k} \right\rangle = \frac{\partial f}{\partial x^k} u^k \frac{\partial}{\partial x^k} = u^k \frac{\partial f}{\partial x^k}
\]

\[
\langle df, u \rangle = \nabla_u f \quad \text{(in the notation of P30.11)}
\]

The covariant derivative extends the directional derivative from scalars to vectors. One introduces from P30.13

\[
\langle dv, u \rangle = \nabla_u v
\]
Like the differential operator $\partial$, the directional derivative operator contains to scalar fields and vector fields. It therefore is just as powerful, its 4 defining properties are given by the following:

**Definition ($\nabla_u V$)**

1. $\nabla_u (V + W) = \nabla_u V + \nabla_u W$ (distributivity)

2. $\nabla_{u + w} V = \nabla_u V + \nabla_w V$

3. For any scalar field $f$

   $\nabla_{fu} V = f \nabla_u V$ (pointwise linearity)

4. $\nabla_u (fV) = \nabla_u (f) V + f \nabla_u V$ (product ("Leibniz") rule)

**Problem (Components of $\nabla$)**

Given: $u = \partial_i u^i$

$V = \partial_i v^i$

Exhibit $\nabla_u V$

**Solution:**

$\nabla_u V = \nabla_{u^k} V$

$= \nabla_{\partial_i} (\partial_i u^k v^i) u^k$

$= \partial_i \partial_j (\partial_i u^k v^i) u^k$

$= \partial_i \partial_j (\partial_i u^k v^i) u^k$

$\nabla_u V = \partial_i (\partial_i u^k v^i) u^k$

$\nabla_u V = \partial_i v^i \partial_i u^k$

Here $v^i \partial_i$ are the components of $\nabla$. 