

Lecture 30

- A) Parallelism [MTW sects 8.3, 8.5, 10.2, 10.4; Box 10.2, 10.3; Fig 10.2]
- B) The covariant differential of a vector field
[MTW sect. 14.5]
- C) The covariant derivative. [MTW § 8.5,
10.2, 10.3, 10.4]
- [In MTW Read Chapter 10]

A) PARALLELISM

30.09

Parallelism is a concept which establishes

$V_{x^a}(x^b) \in P$

a one-to-one relation between vectors in tangent

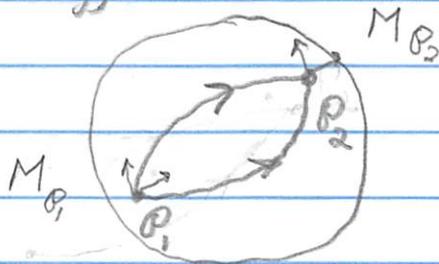
INSERT a footnote on page 30.16

spaces at different points of a manifold,

points which can be connected by curves

between them. These curves are in

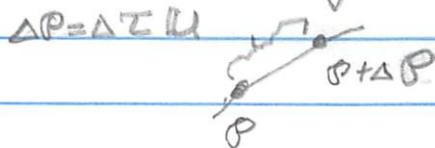
general different



and thus give rise to different isomorphisms between the tangent spaces.

However, for infinitesimal close points

the connecting curves are unique,



are determined by its tangent vector, and are given by the relation linear in the curve parameter τ :

$$\Delta P = \Delta \tau U; \{ \Delta X^R = \Delta \tau U^R \}$$

parameter τ

INSERT a footnote (for page 30.1a)

\begin{footnote}

It is understood but is worth mentioning that the concept "parallelism" is applicable to vector spaces other than tangent spaces attached to the points of a manifold. Aside from differentiability restrictions the only restriction is that these vector spaces be copies of one and the same vector space, even if it is a complex vector space, e.g. the wave mechanical state space of a particle at each point.

\end{footnote}

The parallel-transport-given isomorphism between M_p and $M_{p+\Delta p}$ is condensed into the vectorial differentials

$$de_i = e_j \otimes \omega_{ij}^j \quad i=1, \dots, n$$

Here the connection one forms

$$\omega_{ij}^j = \Gamma_{ik}^j(p) dx^k$$

is given in terms of the functions

$$\Gamma_{ik}^j(p) = \Gamma_{ik}^j(x^1, \dots, x^n),$$

the "Christoffel symbols of the 2nd kind."

THE COVARIANT DIFFERENTIAL OF A VECTOR FIELD

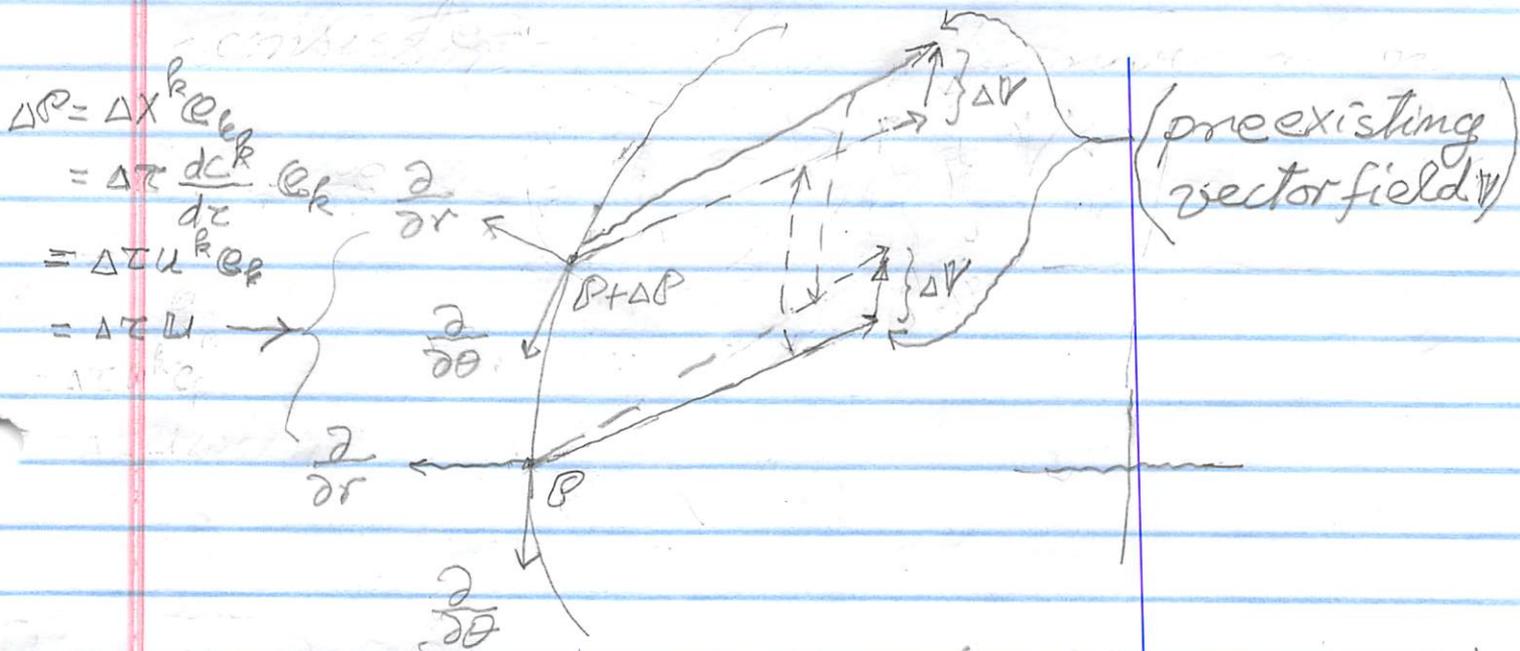
Q: Given that de_i mathematized

the rate of change of the basis

vectors $e_i = \frac{\partial}{\partial x^i}$ in each M_p , $i=1, \dots, n$

how does one apply this rate of change to
a vector field

$$V \Rightarrow v^i(x^1, \dots, x^n) e_i \text{??}$$



$$\Delta V = \langle dV, \Delta \tau u \rangle = V(P+\Delta P) - (\text{transport of } V \text{ from } M_P \text{ to } M_{P+\Delta P})$$

[Here V is a linear combination of basis fields $\{e_i\}$

whose coefficients are scalar fields.]

A: Use the rule for taking the differential of a product.

The rule for taking the differential of a product is based on taking its difference

$$\Delta(e_i v^i)$$

on a small scale ("infinitesimal displacement"). For small displacements Δe_i goes linearly with the separation; doubling the separation will double Δe_i . The same thing holds for each of the Δv^i , consequently

$$\begin{aligned} & (e_i + \Delta e_i)(v^i + \Delta v^i) - e_i v^i = \\ & = e_i \Delta v^i + \Delta e_i v^i + \text{negligible terms} \end{aligned}$$

The law of parallel transport is a one-to-one and onto relation between M_p and its neighboring vector space $M_{p+\Delta p}$, a relation that holds for all vectors in $M_{p+\Delta p}$, including e_i and $v = e_i v^i$. The Schild ladder construction bears this out.

Thus one has not only

$$\begin{aligned}\Delta e_i &= e_i(p+\Delta p) - (\parallel \text{xport of } e_i \text{ from } p \text{ to } p+\Delta p) \\ &= e_i(p) - (\parallel \text{xport of } e_i \text{ from } p+\Delta p \text{ to } p)\end{aligned}$$

The same Schild ladder construction holds for

$$\begin{aligned}\Delta v &= v(p+\Delta p) - (\parallel \text{xport of } v \text{ from } p \text{ to } p+\Delta p) \\ &= v(p) - (\parallel \text{xport of } v \text{ from } p+\Delta p \text{ to } p)\end{aligned}$$

30, 2d

So with the understanding that ΔV
and Δe_i are obtained from the
Schild ladder constructions of V and e_i
subtracted respectively from the
preexisting V and e_i , one
obtains.

30.2e

$$\Delta V = V(P + \Delta P) - (\text{transport of } V \text{ from } P \text{ to } P + \Delta P)$$

$$= \Delta(e_i v^i)$$

$$= e_i \Delta v^i + \Delta e_i v^i$$

$$= e_i \left\langle dv^i, \Delta \tau u^k \frac{\partial}{\partial x^k} \right\rangle + \left\langle e_j \omega^j_i v^i, \Delta \tau u^k \frac{\partial}{\partial x^k} \right\rangle$$

$$= e_i \frac{\partial v^i}{\partial x^k} u^k \Delta \tau + e_j \Gamma^j_{ik} v^i u^k \Delta \tau$$

i → j

$$= \left\langle e_j \otimes \left(\frac{\partial v^j}{\partial x^k} + \Gamma^j_{ik} v^i \right) dx^k, \underbrace{\Delta \tau u^k}_{\Delta \tau u} e_k \right\rangle$$

$$\therefore \underline{dV} = e_j \otimes \underbrace{\left(\frac{\partial v^j}{\partial x^k} + \Gamma^j_{ik} v^i \right)}_{v^j_{;k}} dx^k$$

$$(\star) \quad \boxed{dV = e_j v^j_{;k} dx^k} \quad \text{where } v^j_{;k} = v^j_{,k} + \Gamma^j_{ik} v^i$$

$$dV = \dots$$

$$|dV| = \dots \quad (*)$$

$$= \dots + \dots$$

dV = rate of change (relative to parallel transport) due to motion into an as-yet-unspecified direction,
 = vector-valued one form

THE COVARIANT DERIVATIVE OF A VECTOR FIELD

Recall (I.) For any scalar f evaluated on a curve $c^i(\tau)$ with tangent

$$\frac{dc^i}{d\tau} \frac{\partial}{\partial x^i} = u^i \frac{\partial}{\partial x^i} = u \quad \left(= D_u \text{ old standard notation} \right)$$

$$\lim_{\Delta\tau \rightarrow 0} \frac{\Delta f}{\Delta\tau} = \lim_{\Delta\tau \rightarrow 0} \frac{f(c^i(\tau+\Delta\tau)) - f(c^i(\tau))}{\Delta\tau} = u^i \frac{\partial f}{\partial x^i} = \nabla_u f$$

$$= \left\langle \frac{\partial f}{\partial x^i} dx^i, \underbrace{u^i \frac{\partial}{\partial x^i}}_{u = D_u} \right\rangle$$

$$= \langle df, u \rangle$$

Thus we have

$$(**) \quad \langle df, u \rangle = \nabla_u f \quad \left(= \text{direction derivative of } f \right)$$

$$df = \nabla f \quad \left(\text{derivative into an as-yet-unspecified direction} \right)$$

Also recall, $\nabla_{\mu} e_i$

II.) $d e_i =$ rate of change (relative to \parallel parallel transport) of e_i into an as-yet-unspecified direction

$=$ parallel transport induced

vector valued 1-form

Consider the rate of change (relative to \parallel parallel transport) into the direction u ;

$$\nabla_u e_i \equiv \lim_{\Delta\tau \rightarrow 0} \frac{\Delta e_i}{\Delta\tau} = \langle d e_i, u \rangle = \lim_{\Delta\tau \rightarrow 0} \frac{e_i(\delta_{ij}^i + \omega_{ij}^k(\Delta\tau u^k)) - e_i}{\Delta\tau}$$

$$= e_j \Gamma_{ik}^j u^k$$

(Directional derivative of e_i)

Thus we

$$\langle d e_i, u \rangle \equiv \nabla_u e_i = e_j \Gamma_{ik}^j u^k \quad (***)$$

and

$$d e_i = \nabla e_i$$

(Derivative of e_i into an as-yet-unspecified direction)

Evaluating $Eqs(*)$ (on page 30.3) on the tangent vector u ,
 applying $Eq.(**)$ (on page 30.4) to each of the functions v^i , $i=1, \dots, n$
 and using $Eq.(***)$ (on page 30.5) one obtains

$$\begin{aligned}
 \text{III,)} \quad \langle dv, u \rangle &\stackrel{(*) \text{ on page 30.2e}}{=} \langle e_i \otimes dv^i + v^i de_i, u^k \frac{\partial}{\partial x^k} \rangle \\
 &= e_i \langle dv^i, u \rangle + v^i \langle de_i, u \rangle \\
 &\quad \downarrow (***) \text{ P30.4} \quad \quad \downarrow (***) \text{ P30.5} \\
 &= e_i \nabla_u v^i + v^i \nabla_u e_i \\
 &\quad \downarrow \text{product rule} \\
 &\equiv \nabla_u (e_i v^i) \quad \left(\text{Directional derivative of } v = e_i v^i \right) \\
 \text{explicitly} \quad &= e_j \left(\frac{\partial v^j}{\partial x^k} u^k + \Gamma_{ik}^j v^i \right) u^k \\
 &= e_j \dot{v}^j_{,k} u^k
 \end{aligned}$$

Thus we have introduced the following directional derivatives

$$\begin{aligned}
 \langle df, u \rangle &= \nabla_u f \quad (\equiv u(f)) \\
 \langle de_i, u \rangle &= \nabla_u e_i = e_j \Gamma_{ik}^j u^k \\
 \langle dv, u \rangle &= \nabla_u v
 \end{aligned}$$

Like the differential operator d , the direction derivative operator pertains to ^{both} scalar fields and vector fields, it therefore is just as powerful. Its 4 defining properties are give by the following

Definition ($\nabla_u V$)

- (1) $\nabla_u (V_1 + V_2) = \nabla_u V_1 + \nabla_u V_2$ (distributivity)
- (2) $\nabla_{u_1 + u_2} V = \nabla_{u_1} V + \nabla_{u_2} V$ "
- (3) For any scalar field f

$$\nabla_{fu} V = f \nabla_u V$$
 (point wise linearity)
- (4) $\nabla_u (fV) = \nabla_u(f) V + f \nabla_u V$ (product ("Leibnitz") rule)

Problem (Components of ∇)

compute $\nabla_{e_k} e_j$.

Solution:

$$\nabla_{e_k} e_j = \langle de_j, e_k \rangle$$

$$= \langle e_j \Gamma_{ik}^j dx^i, \frac{\partial}{\partial x^k} \rangle$$

$$= e_j \Gamma_{ik}^j \langle dx^i, \frac{\partial}{\partial x^k} \rangle$$

$$= e_j \Gamma_{ik}^j$$

"Christoffel symbol of the 2nd kind"

Problem (components of ∇)

30.9

Given: $u = e_R u^k$
 $v = e_i v^i$

Exhibit $\nabla_u v$

Solution:

$$\nabla_u v = \nabla_{e_R} u^k v$$

$$= \nabla_{e_R} (e_i v^i) u^k$$

$$= \left[e_i \nabla_{e_R} v^i + \nabla_{e_R} (e_i) v^i \right] u^k$$

$$= \left[e_i \frac{\partial v^i}{\partial x^k} + e_j \Gamma_{ik}^j v^i \right] u^k$$

$$\nabla_u v = e_i \left[\frac{\partial v^i}{\partial x^k} + \Gamma_{jk}^i v^j \right] u^k$$

$$\nabla_u v = e_i \cdot v^i{}_{;k} u^k$$

Here $v^i{}_{;k}$ are the components of ∇ .