

LECTURE 32

[MTW §10.4]

1. Covariant derivative of a 1-form (covector field)
2. Commutator vs Covariant Derivative

Pointwise linearity as the signature

of a tensor map [MTW Box 10.3B]

3. Parallel vector fields

4. The torsion tensor.

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Having become familiar with the covariant derivatives $\nabla_{e_i} e_j = e_k \Gamma_{ji}^k$ of the basis vectors, let us apply them and the four properties to compute the derivative $\nabla_u V$ and $\nabla_u \Omega$ of the vector V and the covector Ω .

Proof:

1. Derivative of a vector V (already done) 32.2
 on page 30, 18

It is easy to evaluate $\nabla_u V$ once we know

$$\nabla_{e_i} e_j = e_k \Gamma_{ji}^k$$

Reminder

Side remark: (This we recall a little bit of) the vector-valued 1-form d_e ; for the vector e_i :

$$\begin{aligned} \text{Notation: } \nabla_{e_i} e_j &= \langle d_e e_j | e_i \rangle = e_k \langle \omega_{ji}^k | e_i \rangle \\ &= e_k \Gamma_{ji}^k \langle \omega^k | e_i \rangle \\ &= e_k \Gamma_{ji}^k \end{aligned}$$

↓

$$\begin{aligned} \nabla_u V &= \nabla_u v^i e_i = \\ &= u^i \left\{ \nabla_{e_i} (v^j) e_j + v^j \nabla_{e_i} (e_j) \right\} \\ &= u^i \left\{ \nabla_{e_i} (v^j) e_j + v^j \nabla_{e_i} e_j \right\} u^k \Gamma_{ji}^k \\ &= u^i \left\{ \frac{\partial v^j}{\partial x^i} e_j + v^j \Gamma_{ji}^k e_k \right\}. \end{aligned}$$

Here $v^j_{,i} = \frac{\partial v^j}{\partial x^i}$ relative to a coordinate basis.

Now change the dummy summation index and obtain

$$= u^i \left\{ v^k_{,i} + v^j \Gamma_{ji}^k \right\} e_k$$

$$\boxed{\nabla_u V = u^i v^k_{,i} e_k} = \langle dV | u \rangle$$

$$= \langle v^k_{,i} i dx^i | u^i \frac{\partial}{\partial x^i} \rangle$$

which is also the vector-valued differential dV evaluated on u :

$$\nabla_u V = \langle dV | u \rangle,$$

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2. Covariant derivative of a 1-form

$$\bar{\nabla} = \bar{\Omega}_i^j \omega^i$$

First one finds the covariant derivative of the dual basis 1-forms ω^i . This duality,

$$\langle \omega^i, e_j \rangle = \delta_j^i$$

holds in every tangent space throughout the manifold. To make sure that the law of parallel transport of vectors does not violate this condition, one must have

$$\begin{aligned} 0 &= \nabla_{e_k} (\delta_j^i) = \nabla_{e_k} \langle \omega^i, e_j \rangle \\ &= \langle \nabla_{e_k} \omega^i, e_j \rangle + \langle \omega^i, \nabla_{e_k} e_j \rangle \end{aligned}$$

$$\text{Thus } \langle \nabla_{e_k} \omega^i, e_j \rangle = -\Gamma_{jk}^i.$$

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The spanning property of the basis $\{e_i\}$ implies that

$$\begin{aligned} \nabla_{e_k} \omega^i &= \langle \nabla_{e_k} \omega^i, e_j \rangle \omega^j \\ &= -\Gamma_{jk}^i \omega^j \end{aligned}$$

Second one extends this result to a general 1-form by using the product rule,

$$\begin{aligned} \nabla_{e_k} (\bar{\Omega}_i^j \omega^i) &= (\nabla_{e_k} \bar{\Omega}_i^j) \omega^i + \bar{\Omega}_i^j \nabla_{e_k} \omega^i \\ &= \frac{\partial \bar{\Omega}_i^j}{\partial x^k} \omega^i - \bar{\Omega}_j^i \Gamma_{ik}^j \omega^i \\ &= \bar{\Omega}_{ijk} \omega^i \end{aligned}$$

Thus one has the covariant derivative of $\bar{\Omega}_i^j$

$$\nabla_{e_k} \bar{\Omega}_i^j = \bar{\Omega}_{ijk} \omega^i$$

where

$$\bar{\Omega}_{ijk} = \frac{\partial \bar{\Omega}_i^j}{\partial x^k} - \bar{\Omega}_j^i \Gamma_{ik}^j$$

are its coordinate components, which is to be compared with $\nabla_{e_k}^i = \frac{\partial v^i}{\partial x^k} + v^i \Gamma_{jk}^i$ for $v = e, \omega^i$.

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3. Comparison between $\nabla_{\mathbf{v}} \mathbf{V}$ and $[\mathbf{V}, \mathbf{U}]$.

a) In what sense is the covariant derivative a tensor? Answer:

(i) Given a fixed vector \mathbf{V} , then

$$\nabla \mathbf{V} : M_\phi \rightarrow M_\phi \\ u \mapsto \nabla_u \mathbf{V}$$

is a tensor because $\nabla \mathbf{V}$ is pointwise linear:

$$f \mathbf{u}_1 + g \mathbf{u}_2 \mapsto f \nabla_{\mathbf{u}_1} \mathbf{V} + g \nabla_{\mathbf{u}_2} \mathbf{V}.$$

Thus $\boxed{\nabla \mathbf{V} \text{ is a tensor}}$ at each point.

b) By contrast consider the Lie derivative

$$\mathcal{L}_{\mathbf{V}} : M_\phi \rightarrow M_\phi \\ u \mapsto \mathcal{L}_{\mathbf{V}} u = [\mathbf{V}, \mathbf{u}],$$

which is not a tensor map because

$$f \mathbf{u} \mapsto [\mathbf{V}, f \mathbf{u}] = f [\mathbf{V}, \mathbf{u}] + \mathbf{V}(f) \mathbf{u} \\ (= f \mathcal{L}_{\mathbf{V}} \mathbf{u} + u \mathcal{L}_{\mathbf{V}}(f))$$

is not pointwise linear.

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Parallel Vector Field

Once we are given or have determined the covariant derivative $\nabla_{e_i} e_j$, or Γ^j_{ik} , or equivalently $d e_i = e_j \otimes \omega^j{}_i$,

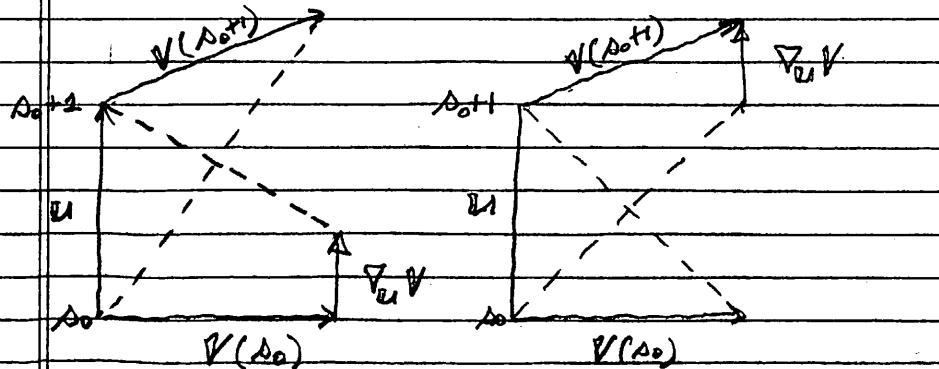
one can construct parallel vector fields.

Such vector fields differ from those,

say v , which are non-parallel along a given direction u , i.e. those which

satisfy

$$\nabla_u v \neq 0$$



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But if a vector field, say

$$v = e_i y^i$$

is parallel under transport along the direction

$$u = e_k y^k$$

then

$$\nabla_u v = 0$$

Thus one must solve the following system of ordinary differential

eq's

GIVEN: a) $u = e_k u^k(x)$

b) One of its integral curves

$$c(s) : \{c^k(s)\}$$

$$\frac{dc}{ds} = e_k \frac{du^k}{ds} = u^k(c(s)) = u^k(s)$$

$$c) v$$

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SOLVE: $\nabla_{\mathbf{v}} \mathbf{V} = \mathbf{0}$ for $\mathbf{V}(s)$ where $\mathbf{u} = \mathbf{e}_2 \mathbf{U}^k(s)$
and

$$\mathbf{y}(s=s_0) = \mathbf{v}_0$$

is the initial vector at $c^k(s)$ of the given curve $c^k(s)$.

SOLUTION:

$$0 = \nabla_{\mathbf{v}} \mathbf{V} = \mathbf{e}_2 \left(\mathbf{U}^k \frac{\partial \mathbf{y}^i}{\partial x^k} + \mathbf{y}^j \Gamma_{jk}^i \mathbf{U}^k \right)$$

refers to the following linear system

of coupled o.d.e.'s for $\mathbf{y}^i(s)$

$$0 = \frac{dy^i}{ds} + y^j \Gamma_{jk}^i(c^k(s)) \mathbf{U}^k$$

We know from the existence and uniqueness

theorem of Lecture 25 that this system

has a unique solution $\{\mathbf{y}^i(s)\}$ for the

specified initial condition

$$\mathbf{y}^i(s=s_0) = \mathbf{v}_0^i$$

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The solution

$$\mathbf{V}(s) = \mathbf{e}_2 \mathbf{y}^i(s)$$

to the differential equation $\nabla_{\mathbf{v}} \mathbf{V} = \mathbf{0}$

is a vector field along the given curve

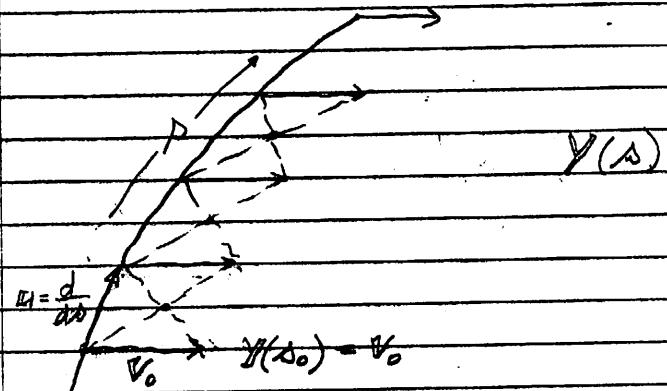
 $\{c^k(s)\}$ whose tangent is $\{\mathbf{U}^k = \frac{dc^k}{ds}\}$:

Figure 32.1 Vector field $\mathbf{V}(s)$ parallel to \mathbf{v}_0 along a given curve $c(s)$

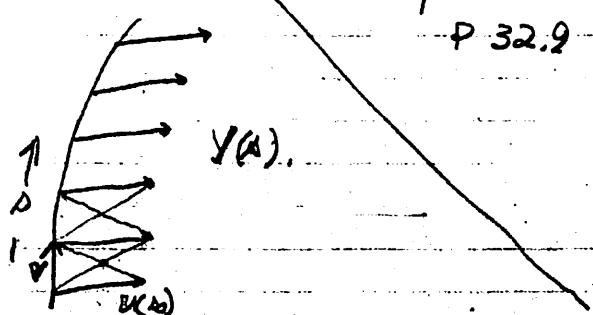
whose tangent is $\mathbf{U}^k = \frac{dc^k}{ds}$

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and the initial conditions are

$$Y(s=s_0) = U(s_0)$$

The solution to this differential equation determines a vector field $Y(s)$ along the given curve $s(t)$ (whose tangent is $v^i = \frac{ds^i}{ds}$)



Superceded by

P 32.9

CLASSIFICATION of PARALLEL TRANSPORT.

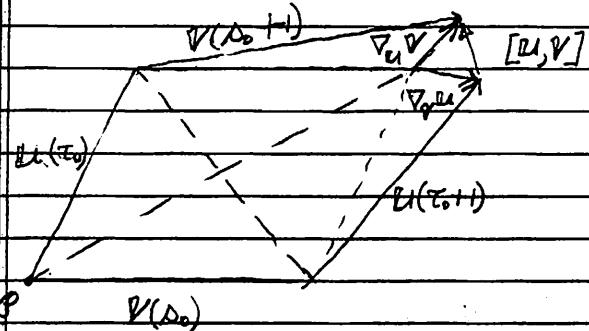
Are there any geometrical restrictions that a given parallel transport law must satisfy?

One reasonable restriction is that a parallelogram be a closed figure.

This can be expressed in terms of the covariant derivatives along two given

53.2

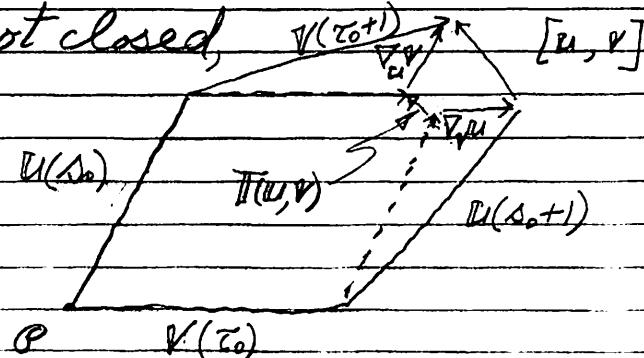
vector fields u and v as follows:



The consistency condition of a parallelogram consisting of a closed figure is expressed by the equation

$$\nabla_u V - \nabla_v U - [U, V] = T(U, V) = 0$$

If $T(U, V) \neq 0$ the parallelogram is not closed,



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then the parallel transport is said to be non-integrable, or, equivalently, the Certain torsion at P is non-zero.

In continuum mechanics of a imperfect crystal (i.e. one permeated by dislocation), the vector $T(u, v)$ is called the Burger vector.

The mathematically most significant property of the expression

$$T(u, v) = \nabla_u v - \nabla_v u - [u, v]$$

is that it is a tensor map. Indeed, we have to following

33.4

Proposition (Torsion Tensor)

T is pointwise linear:

$$T : M_p \times M_p \rightarrow M_p$$

$$(u, v) \mapsto T(u, v) = \text{above expression}$$

has the property

$$T(fu, v) = f T(u, v)$$

$$T(u, gv) = g T(u, v)$$

at each point $P \quad \forall f, g \in C(M, P, R)$.

Thus T is a tensor at each point P .

Comment.

The consistency of parallel transport (i.e. its integrability) is expressed by the condition that

$$T(u, v) = 0$$

for all vector fields u and v .