LECTURE 33

Torsion tensor (recap)

Cartan's 1st structural Eq'n

The 1-2 Version of Stokes' Thm.
[MTW sect. 3]

The Infinitesimal Stokes' Thm

[MTW sect 14.5, S&T sect 7]
Let \( x^k(A, x) \) be the 2-D domain spanned by the integral curves whose tangents are \( \Delta x \) and \( \Delta y \):

\[ P = \{ x^k(A, x) \} \]

\[ U = \frac{\partial x^k}{\partial x} \Delta x \]

\[ V = \frac{\partial x^k}{\partial y} \Delta y \]

\[ T(U, V) = \Delta x \Delta y \left( \nabla U - \nabla V - [U, V] \right) \]

Note: In the diagram in class, the parameter increments \( \Delta x \) and \( \Delta y \) were absorbed into the vectors \( U \) and \( V \):

\[ U = \Delta x \frac{\partial x^k}{\partial x} \frac{\partial}{\partial x^k} \] so that \( \delta P = U \) and

\[ V = \Delta y \frac{\partial x^k}{\partial y} \frac{\partial}{\partial x^k} \]
1) Cartan's 1st structural equation

The mapping

\[ \Pi : M_0 \times M_0 \rightarrow M_0 \]

\[ (\alpha, \nu) \mapsto \Pi(\alpha, \nu) = \nabla_{\nu} \alpha - \nabla_{\alpha} \nu - [\nu, \alpha] \]

is pointwise linear and hence is a tensor map, i.e., a multilinear map. A tensor for each tangent space \( M_0 \).
1) Cartan's 1st Structural Equation

Being a tensor map,

\[ T(u, v) = \nabla_u v - \nabla_v u - [u, v] \]

can be expanded in terms of the vector basis and its dual,

\[ T = e^k T^k_{\ n} \omega^m \otimes \omega^n; \quad T(u, v) = e^k T^k_{\ mn} \omega^m(u) \omega^n(v) \]

with the expansion coefficients \( T^k_{\ mn} \) to be determined. By expressing \( T \) in terms of the connection 1-forms \( \omega^i \), one obtains Cartan's 1st structure equation of differential geometry as follows:
Choose a field of basis vector $\xi e_i$ and its dual $\xi e_i$:

$$\langle \omega^i, e_i \rangle = \delta_i^i.$$ 

Expand the two given vector fields $u$ and $v$ in $T(u, v)$ in terms of these basis elements:

$$u = e_i \langle \omega^i, u \rangle = e_i u^i(x),$$

$$v = e_i \langle \omega^i, v \rangle = e_i v^i(x).$$

Using the product rule one obtains

$$\nabla_u v - \nabla_v u - [u, v] = T(u, v) =$$

$$e_i \nabla_u \langle \omega^i, v \rangle - e_i \nabla_v \langle \omega^i, u \rangle - e_i \langle \omega^i, [u, v] \rangle$$

$$+ (\nabla_v e_i) \langle \omega^i, v \rangle - (\nabla_u e_i) \langle \omega^i, u \rangle$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$e_i \langle \omega^i, u \rangle \quad e_i \langle \omega^i, v \rangle \quad \text{Law of parallel transport}\}$$

Letting $i = i', j = i''$ then dropping the prime ("index gymnastics") and...
recalling that

\[ \nabla_u \langle \omega^2, v \rangle = u \langle \omega^2, v \rangle (= D_u v^2) \]

\[ \nabla_v \langle \omega^2, u \rangle = v \langle \omega^2, u \rangle (= D_v u^2) \]

are directional derivatives of scalars one finds

\[ T(u, v) = \]

\[ = e_i \left\{ \left[ u \langle \omega^2, v \rangle - v \langle \omega^2, u \rangle - \langle \omega^2, [u, v] \rangle \right] \right\} \]

\[ \int \omega = \left[ \int \omega^2 \cdot u \times v \right] \quad \text{"infinitesimal Stokes Thm"} \]

\[ + \langle \omega^2 \cdot \omega^j \times \omega^i \cdot \omega^i \times \omega^j, u \times v \rangle \}

\[ = e_i \cdot \langle dw^2 + \omega^2 \cdot \omega^j \times \omega^i, u \times v \rangle \quad \text{vector valued 2-form} \]

This holds \( \forall \) pairs of vector fields \( u \times v \)

Thus the explicit expression for the map \( T \) is

\[ T = e_i \otimes (dw^i + \omega^2 \cdot \omega^j \times \omega^i) = e_i \otimes \Omega^2 \]
This is Cartan's 1st structural equation. It is an explicit expression for the tensor map $T$, Cartan's torsion tensor:

$$T = e_i \otimes \Omega^i,$$

The two-form

$$\Omega^i = d\omega^i + \omega^j \wedge \omega^i$$

is called Cartan's torsion 2-form.

By expanding $d\omega^i$ and $\omega^j \wedge \omega^i$ in terms of $\omega^m \wedge \omega^n$ one obtains

$$T = \frac{1}{2} e_i T^i_{mn} \omega^m \wedge \omega^n = e_i T^i_{mn} \omega^m \wedge \omega^n$$

$$\Omega^i = \frac{1}{2} T^i_{mn} \omega^m \wedge \omega^n$$

The parallel transport is said to be integrable if Cartan's torsion vectorial 2-form vanishes:

$$e_i \otimes \Omega^i = 0$$
II Stokes Theorem

Consider the line of a one-dimensional form $\omega = \int\alpha \, d\gamma(\alpha)$ around a closed loop determined by the vector fields $\mathbf{u}$ and $\mathbf{v}$.

\[ \oint \omega = \int_{\mathbf{u}} \alpha \, ds + \frac{1}{2} \int_{\mathbf{u} \times \mathbf{v}} \alpha \, ds + \int_{\mathbf{u}} \left( -\mathbf{V} \cdot \nabla \right) d\mathbf{u} \]

Applying the mean value theorem for integrals whose limits are close together, one finds to second order accuracy that...
\[ \delta \omega = \langle \omega, \nu \rangle \Delta \Phi - \langle \omega, \nu \rangle \Delta \Phi + \langle \omega, \nu \rangle \Delta \Phi - \langle \omega, \nu \rangle \Delta \Phi \]

Hence

\[ \Delta \nu = \frac{\langle \omega, \nu \rangle \Delta \Phi}{\langle \omega, \nu \rangle} \]

Q: What is the value of the three term expression inside the curly bracket?

A: The answer, to be validated below, is that

\[ u \langle \omega, \nu \rangle - v \langle \omega, \nu \rangle - \langle \omega, [\nu, \omega] \rangle = \langle d\omega, u \times \nu \rangle \]

Consequently, the line integral over the boundary \( \partial A \) of the area \( A \) spanned by the vectors \( d\omega \) and \( d\omega \times \nu \) is

\[ \oint_{\partial A} \omega = \int_{\partial A} \langle d\omega, u \times \nu \rangle d\sigma = \int_{\partial A} \langle d\omega, u \times \nu \rangle d\sigma 
\]
This is the infinitesimal version of the 1-2 version of Stokes' theorem, which relates the area integral over a region spanned by \( \partial S \) and \( \partial \Gamma \) to the line integral over the boundary of that region.

It is parametrized by the curve parameter \( t \) and \( \xi \),

\[
X^k(\xi, t) \quad \text{for} \quad 0 < \Delta \xi \leq 5 \\
2 \Delta \xi \leq t \\
2 \Delta \xi \leq 2 \Delta t
\]

so that

\[
v = \frac{\partial X^k}{\partial \xi} \frac{\partial}{\partial x^k} \\
v = \frac{\partial X^k}{\partial t} \frac{\partial}{\partial x^k}
\]

are the given tangent. One needs to evaluate \( \langle dw, u \times v \rangle \). To this end one observes that the 2-form \( dw \)
is given to by
\[ d\omega = d\left(\omega^i dx^i\right) = \frac{\partial \omega^i}{\partial x^j} dx^j dx^i \]

Consequently
\[ \langle d\omega, u \times v \rangle = \left(\frac{\partial \omega^i}{\partial x^j} - \frac{\partial \omega^i}{\partial x^j}\right) \langle dx^j \circ dx^i, \frac{\partial x^k}{\partial x^j} \frac{\partial x^l}{\partial x^i} \frac{\partial x^m}{\partial x^k} \frac{\partial x^n}{\partial x^l} \rangle \]

\[ = \left(\frac{\partial \omega^i}{\partial x^j} - \frac{\partial \omega^i}{\partial x^j}\right) \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^i} \frac{\partial x^k}{\partial x^l} \frac{\partial x^l}{\partial x^k} \frac{\partial x^m}{\partial x^k} \frac{\partial x^n}{\partial x^l} = 0 \]
Combining Eqs. (1), (2), and (3) on p 33.6, 33.8

\[ \oint \omega = \langle dw, u \times v \rangle \Delta S \Delta z \]

\[ - \int_{SS} \left( \frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y} \right) \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \, dx \, dy \]

The right hand side is a surface integral over an area parametrized by \( s \) and \( t \).

This result, written symbolically as

\[ \oint \omega = \iiint_{\text{area}} \omega \, \text{d}(\text{area}) \]

is known as the 1-2 version of Stokes' theorem. Applied to an infinitesimal area, it is a mere restatement of Eq. (2) on page 33.6.

It therefore is appropriate to refer to
This equation is the "infinitesimal 1-2 version of Stokes' theorem.


The fundamental equality Eq. (2) on page 33.6, namely

$$\langle dw, u \times v \rangle = u(\langle w, v \rangle) - v(\langle w, u \rangle) - \langle w, [u, v] \rangle$$

is a statement which combines linear algebra with calculus. Its validity is based on the observation that both sides of this equation are pointwise linear in $w$. Hence it is enough to validate it for the 1-form

$$w = \gamma \, d \, y$$
Without loss of generality.

Proof: we consider \( w = f \, dg \) so that we have

\[
\begin{align*}
\text{L.H.S.} & = \langle dw, u \times v \rangle = \langle df \wedge dg, u \times v \rangle \\
& = \langle df, u \rangle \langle dg, v \rangle - \langle dg, u \rangle \langle df, v \rangle \\
& = u(f) v(g) - u(g) v(f)
\end{align*}
\]

Here \( u(f), \) etc. have the usual meaning.

\[
\begin{align*}
\text{R.H.S.} & = \langle u, u \rangle v - v \langle u, u \rangle - \langle u, u \rangle |
\end{align*}
\]

\[
\begin{align*}
& = u(f) \langle dg, v \rangle - v(f) \langle dg, u \rangle - f \langle dg, u \rangle (g) \\
& = u(f) v(g) - v(f) u(g) \\
& \quad + f \langle u(v(g)), - f v(u(g)) \rangle - f \{ u(v(g)) - v(u(g)) \} \\
& = u(f) v(g) - v(f) u(g) + \text{canceled terms}
\end{align*}
\]

\[
\begin{align*}
\therefore \text{L.H.S.} & = \text{R.H.S.}
\end{align*}
\]

This validates the boxed Stokes' Thm on page 33.10, and therefore establishes
Cartan's 1st structural equation

\[ \delta^i_j = dw^i + \omega^i_j \wedge \omega_j^i \]