LECTURE 34

Curvature: What does it mean?

Riemann Curvature: Parallel transport around a closed loop

[MTW sect. 11.4; 8.7]

Cartan's 2nd Structural Eq'n

[MTW sect. 14.5]

Components of Riemann relative to a coordinate basis

[MTW sect. 11.5]
Curvature

A given law of parallel transport provides the rules for constructing parallel vectors. The application of this construction to two vectors emanating from the same point yields a parallelogram. However, whether or not the parallelogram is a closed figure depends on the torsion (tensor) of the parallel transport law. The displacement necessary to form a closed figure is expressed by the (vectorial) value of the torsion tensor applied to the two vectors spanning the parallelogram. Thus parallel transport has a displacement aspect to it, and torsion, via parallelograms, records this displacement into a mathematically precise form.

But parallel transport also has a rotation aspect too, and it is revealed by means of the curvature via vectors parallel-transported around a closed curve.

4.1 Rotation of a Vector Transported Around a Closed Loop.

Let us illustrate this rotation on the parallel transport characterized by Euclid's closed parallelogram construction on the two sphere $S^2$.

We take a closed circle of constant latitude $\theta$, and using the closed parallelogram construction, parallel transport a vector around this closed loop.
The vectors are parallel because they are the opposite sides of a succession of parallelograms. It is evident that the successive opposite sides of these parallelograms acquire their parallelness from the familiar parallel transport of the ambient three-dimensional space. In other words, the parallelism of Euclidean three-dimensional space induces a unique parallel transport on the two sphere $S^2$.

Q: What is the result of this transport?
A: The result of this parallel transport can be exhibited quantitatively by cutting out an annular strip surrounding the circle of constant latitude, $\phi$-constant. This annular strip is tangent to the bottom of a cone whose apex angle is $2 \times (90^\circ - \phi)$.

Cut this cone along an edge, and flatten the cone out so that it becomes a disk with a missing sector of angle $\phi$.

- same vector on $S^2$
- radius = $L$

Total circumference = $2\pi L$
(not partial, i.e. meridian circle)
The parallel vector field is along the arc perimeter of the disk. It is evident that after a full circuit of parallel transport, a vector gets rotated by the angle $\alpha$.

More precisely, we have

$$\alpha = \text{Rotation (angle) of a vector after parallel transport around a meridian circle of latitude away from the North pole}$$

$$\alpha = \frac{\text{arc length of complete circle}}{\text{arc length of meridian circle}} L$$

$$\alpha = \frac{2\pi L - 2\pi L \cdot \sin(90^\circ - \theta)}{L}$$

$$\alpha = 2\pi (1 - \cos \theta)$$

This angle leads us to the following tentative definition:

The curvature permeating an enclosed area yields the angular rotation suffered by a vector parallel transported around the perimeter of that area:

$$\alpha = \text{(curvature)} \times \text{enclosed area}$$

or

$$\text{curvature} = \frac{\alpha}{\int_{S} \frac{\partial}{\partial p} \frac{\theta}{r^2 \sin \theta} \, dS} = \frac{2\pi (1 - \cos \theta)}{2\pi r^2 \sin \theta}$$

The radius $r$ is called the radius of curvature of the enclosed area.
Reminders about parallel transport.
Consider some arbitrarily specified vector field $\mathbf{W}$ along a curve whose tangent vector is $\mathbf{v}$, and which curve is parametrized by $s$.

$W(0)$ and $W(a)$ are two vectors, one at $s=s_1$, the other at $s=0$.

Definition of covariant derivative of $W$.

$$\nabla_v W = \lim_{\Delta s \to 0} \frac{[W(0+\Delta s)]_{parallel\text{transported}} - W(0)}{\Delta s}$$
Definition: A parallel vector field along a curve through $\mathbf{u} \Leftrightarrow \nabla_u \mathbf{u} = 0$

Definition: A curve is called a geodesic if its tangent vector is parallel along the curve $\Leftrightarrow \nabla_\mathbf{u} \mathbf{u} = 0$

5. Curvature: Generator of rotation under parallel transport around a closed curve.

a) GIVEN: A closed (broken) curve determined by two vector fields $\mathbf{u}$ and $\mathbf{v}$

b) (ii) Parallel transport the vector $\mathbf{W}(1)$ back to point $P$ along two different broken paths.

\[ 1 \rightarrow P + \delta P \rightarrow P \]

and

\[ 1 \rightarrow 2 \rightarrow P + \delta P \rightarrow P \]

\[ \mathbf{W}(1) \rightarrow \mathbf{W}(1) \rightarrow \mathbf{W}(1) \rightarrow \mathbf{W}(1) \]
### (iii) The purpose of the vector field $\mathbf{w}$ is to make the parallel transport visible and documentable (in the form of the solid and dashed vectors) along each leg individually of the two broken paths.

- One will find that, even though one uses that intermediate fiducial vector field $\mathbf{w}$ along the two broken paths, once the parallel transport of $\mathbf{w}(1)$ from $1$ to $p$, has been performed along the two paths, the difference between the different results will be independent of the intermediate vector field.

### c) Parallel transport of $\mathbf{w}(1)$ along 1st broken path is achieved by means of a Taylor series along each segment done up to second order accuracy:

- Vector at $p + \delta p$ which is parallel to $\mathbf{w}(1)$ and has been parallel translated from $1$ to $p + \delta p$ + higher order terms

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<th>$\mathbf{w}(1)$</th>
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<tr>
<td>$\mathbf{w}_0 + \delta \mathbf{v}_0 \mathbf{w} + \frac{\delta^2}{2} \mathbf{v}_0 \nabla \mathbf{v}_0 \mathbf{w} + \ldots$</td>
<td>$\mathbf{w}_0 + \delta \mathbf{v}_0 \mathbf{w}(p) + \delta^2 \mathbf{v}_0 \nabla \mathbf{v}_0 \mathbf{w}(p) + \ldots$</td>
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| \(= \text{vector at } P \text{ which is parallel to } W(v) \)  
\(\text{and has been parallel translated along} \)
\(1 \to P + \varepsilon P \to P \). | \(= s, \tau, \text{ Riemann } (\ldots; W, u, v) \)
\(= \text{vector at } P \) |
| 1) Parallel transport of \(W(v)\) along \(2^{nd} \text{ broken path} \)

\(1 \to 2 \to P + \Delta P \to P\). | \(= \text{vectorial amount of rotation that} \)
\(W\) undergoes when translated around the closed path counter-clockwise. |
| yields \( (\text{same as c) on p3411 except } r, W = sW) + A \tau, \nu, \sigma W = \) \(= \text{parallel translate of } W(v) \text{ along} \)
\(1 \to 2 \to P + \Delta P \to P \). | f) \(= \) The result: The curvature operator \(R\) is pointwise linear |
| \(e) \) Parallel translate of \(W(v)\) to point \(P\) \(2^{nd} \text{ broken path} \) \(- (\text{parallel translate of } W(v) \text{ to point } P \text{ along } 1^{st} \text{ broken path}) \) \(- (\text{vector in } c) \) \(- (\text{vector in } d) \) \(= s, \tau, \) \(\left[ \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \right] W + \ldots\) \(= s, \tau, R(u, v) W\) \(= s, \tau, R(u, v) W\) | \(= s, \tau, R(u, v) W\) \(= \) \(= \) Conclusion:
\(R(u, v) W = \left[ \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \right] W\) \(\) is a tensor map. |