LECTURE 35

Cartan's 2nd structural equation

Curvature as a rotation generator

Cartan's derivation of his two structural equation.
Carnot's 2nd Structural Equation:

An explicit representation of the curvature tensor \( \omega \) tensor-valued two-form.

The Riemann curvature tensor is defined by the pointwise linear tensor map

\[
R : M_p \times M_p \times M_p \to M_p
\]

\[
(W, U, V) \mapsto R(\cdots, W, U, V)
\]

\[
= \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]} w
\]

\[
= e_i \left< w^k, \nabla_v \nabla_u \nabla_i w \right>
\]

The explicit representation of this tensor is found by setting

\[
u = e_i \left< w^k, \nabla_i \nabla_k \nabla_j w \right>
\]

\[
v = e_i \left< w^k, \nabla_i \nabla_j w \right>
\]

\[
w = e_i \left< w^k, \nabla_i w \right>
\]

Just as before, we need to calculate the covariant derivatives

\[
\nabla_v (e_i w^k) = \nabla_v (e_i) w^k + e_i \nabla_v (w^k)
\]

\[
= e_i \left< w^l, \nabla_v \nabla_l \nabla_i w \right> + e_i \nabla_v \nabla_i w
\]

\[
= e_i \left< w^l, \nabla_v \nabla_l \nabla_i w \right>
\]

\[
= e_i \left< w^l, \nabla_v \nabla_l \nabla_i w \right> + e_i \nabla_v \nabla_i w
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= e_i \left< w^l, \nabla_v \nabla_l \nabla_i w \right> + e_i \nabla_v \nabla_i w
\]

The expression for the curvature tensor is found on page 34.13.
Aside from the terms linear in \( w^k \), we have its first derivative terms
\[
e_\mathbf{k} \langle w^k, U \rangle U(w^k) + e_\mathbf{k} \langle \omega^k, U \rangle \nabla (w^k)
- e_\mathbf{k} \langle w^k, U \rangle V(w^k) - e_\mathbf{k} \langle \omega^k, V \rangle U(w^k)
= 0,
\]
which cancel, as do the second derivative terms
\[
e_\mathbf{k} \nu(U(w^k)) - e_\mathbf{k} \nu(U(V(w^k))) - e_\mathbf{k} \nu(V(w^k)) = 0.
\]
Thus we are left with
\[
1 - \varnothing - \varnothing = e_\mathbf{j} \langle \omega^k, \omega^k, U \times V \rangle - e_\mathbf{j} \langle \omega^k, \omega^k, V \times U \rangle
+ e_\mathbf{k} \langle \omega^k, V \rangle - e_\mathbf{k} \langle V, \omega^k \rangle - e_\mathbf{k} \langle \omega^k, U \rangle
\]

According to the infinitesimal version of Stokes' theorem on page 33,10 we have
\[
e_\mathbf{j} \langle \omega^k, U \times V \rangle
\]
consequently
\[
1 - \varnothing - \varnothing = e_\mathbf{j} \langle \omega^k, \omega^k, U \times V \rangle - e_\mathbf{j} \langle \omega^k, \omega^k, V \times U \rangle + d_\mathbf{k} \langle \omega^k, U \times V \rangle w^k.
\]

Thus the amount of vectorial change in \( U \) resulting from its parallel transport around the quadrilateral spanned by \( U \times V \)
\[
= e_\mathbf{j} \otimes \omega^k \langle d \omega^k, e^k + \omega^k \wedge e^k, e^k \rangle
= e_\mathbf{j} \otimes \omega^k \langle d \omega^k, e^k + \omega^k \wedge e^k, e^k \rangle
= e_\mathbf{j} \otimes \omega^k \Rightarrow \nabla^k \cdot (\omega^k U \times V)
\]
Here
\[
\nabla^k \cdot w = dw^k + \omega^k \wedge w^k = R^k \wedge \rho^k \wedge w^k
\]
is the curvature 2-form, which in
\[
\mathbb{R}^n = e_\mathbf{j} \otimes \omega^k \otimes \rho^k = e_\mathbf{j} \otimes \omega^k \Rightarrow R^k \wedge \rho^k \wedge w^k
\]
is the Riemann curvature tensor \( R(\cdots) \).
MTW's notation is given as
\[ R^m_{\;\alpha} = d\omega^m_{\;\alpha} + \omega^m_{\;\beta} \wedge \omega^\beta_{\;\alpha} \] (\( = R^m_{\;\alpha} + \omega^m_{\;\alpha} \) )

This is Cartan's second structural equation. It is to be distinguished from Cartan's first structural equation on page 33.4:
\[ \Sigma^m_{\;\alpha} = d\omega^m_{\;\alpha} + \omega^m_{\;\beta} \wedge \omega^\beta_{\;\alpha} \] (\( = T^m_{\;\alpha} + \omega^m_{\;\alpha} \) )

Both equations are 2-forms that characterize the law of parallel transport
\[ d\omega^m_{\;\alpha} = e^m_{\;\nu} \wedge d\omega_{\nu \alpha} \]
which is given by the connection 1-form
\[ \omega^m_{\;\alpha} \]

The two tensors corresponding to the two Cartan structural equations are

The torsion tensor
\[ T = e^m_{\;\alpha} \wedge d\omega^m_{\;\alpha} = e^m_{\;\alpha} (d\omega_{\alpha} + \omega_{\alpha} \wedge \omega_{\beta}) \]
which is a vector-valued 2-form

The curvature tensor
\[ R = e^m_{\;\alpha} \wedge \omega^m_{\;\beta} = e^m_{\;\alpha} \wedge \omega^m_{\;\beta} (d\omega_{\beta} + \omega_{\beta} \wedge \omega_{\gamma}) \]
which is a (\( \Omega \) )-tensor valued 2-form.

Both are 2-forms with the implication that both are to be evaluated on an element of area whose boundary is delimited by vectors U, V, and their commutator \([U, V]\)
A) Torsion Induced Displacement.

Once evaluated on such an element the vector-valued two form yields a vector

\[ \Theta(U) = e_\mu \langle \Omega_{\mu}, u \times v \rangle \]

which is an infinitesimal displacement.

B) Curvature Induced Rotation.

By contrast, evaluation of the (1)-tensor valued curvature 2-form yields

\[ R(U, U, U) = e_\mu \Theta(U) \langle \Omega_{\mu}, u \times v \rangle \]

which is an infinitesimal rotation.

Indeed, upon parallel transporting the vector

\[ W = e_\mu \omega^\mu \]

around an infinitesimal loop determined by \( u \) and \( v \), one finds that this vector points into a new direction as given by the new vector

\[ W_{\text{new}} = e_\mu \omega_\mu^{\text{new}} \]

\[ W + R(W, W, U) = e_\mu \omega^\mu (\delta^\mu_\nu + \langle \Omega^\mu_{\nu}, u \times v \rangle) \]

\[ = e_\mu \omega^\mu_\nu + \langle \Omega^\mu_\nu, u \times v \rangle \]

Thus the curvature in the area spanned by \( u \) and \( v \) manifests itself as the linear transformation.

\[ R(U, U, U) = \langle \Omega_{\mu}, u \times v \rangle \]

\[ \left[ \delta^\mu_\nu + R^{\alpha}_{\mu \nu \alpha \beta} u^\alpha v^\beta \right] \omega_\nu^\beta = \omega_\nu^{\text{new}} \]

The infinitesimal matrix

\[ R^{\alpha}_{\mu \nu \alpha \beta} u^\alpha v^\beta = \langle \Omega^{\alpha}_{\nu}, u \times v \rangle \]

is called the generator of that linear transformation, which differs only infinitesimally from the identity.

Comment: We shall see that whenever the parallel transport is induced from a given metric tensor field then that linear transformation...
preserves the lengths of vectors which

to say that it's a rotation
Cartan's Method of Deriving
His Two Structural Equation

Step 1: Introduce \( dp = e_u \otimes w^m \) a vector-valued differential 1-form by the following process of abstraction applied to a Taylor series:

\[
\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\partial f}{\partial x^m} \Delta x^m + \ldots \quad \text{drop the "\( \Delta x \)" and neglect higher order terms}
\]

\[
= \Delta x^m \frac{\partial f}{\partial x^m}
\]

\[
= \Delta u^m \frac{\partial f}{\partial x^m}
\]

\[
= \Delta u^m \left( f \right) \quad \text{"change in position" due to displacement}
\]

\[
\Delta p = \Delta u^m \left\{ \Delta x^m = \Delta u^m \right\}
\]

\[
= \Delta \Delta u^m \left\{ \Delta x^m = \Delta u^m \right\}
\]

\[
= \Delta \Delta u^m \left\{ \Delta x^m = \Delta u^m \right\}
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\]

Hence introduce

\[
dp = e_u \otimes w^m = \"\text{change in an as-yet-unspecified position due to an as-yet-unspecified displacement}\"
\]

Cartan calls this simply a displacement vector.

However, post WWII mathematicians (like Claude Chevalley) observed that

\[
dp(v) = e_u \langle w^m, v \rangle = e_u v^m
\]

\[
= v
\]

Thus

\[
dp = e_u \otimes w^m
\]

is a vector-valued differential 1-form, a tensor of rank (1). For valid reasons it is also called Cartan's unit tensor.
Step 2. Apply the exterior derivative to $d\omega$ and obtain Cartan's torsion tensor:

\[ d\omega = d(e_\mu \omega^\mu) \]

\[ = e_\mu \, dw^\mu + de_\mu \wedge dw^\mu \]

\[ = e_\mu \, dw^\mu + e_\mu \omega^\nu \wedge dw^\nu \text{ (Law of Superposition)} \]

\[ = e_\mu \left( dw^\mu + \omega^\nu \wedge dw^\nu \right) = e_\mu \wedge dw^\mu \]

This is Cartan's 1st structural equation, a (1) tensor valued 2-form.

Comment:

Note that the tensor product sign $\otimes$ in step 2 and in step 3 is totally unnecessary, provided the multiplicative vector coefficients of the vector-valued differential forms always remain on the left of differential forms proper.

Step 3. Apply the exterior derivative to the law of parallel transport:

\[ d\gamma^\mu = e_\nu \otimes \gamma^\mu, \text{ and obtain the curvature tensor:} \]

\[ dde_\mu = d(e_\nu \otimes \gamma^\mu) \]

\[ = e_\nu \, dw^\nu + de_\nu \wedge dw^\nu \]

\[ = e_\nu \, dw^\nu + e_\nu \omega^\nu \wedge dw^\nu \]

\[ = e_\nu \left( dw^\nu + \omega^\nu \wedge dw^\nu \right) = e_\nu \wedge dw^\nu \]