

## LECTURE 35

Cartan's 2<sup>nd</sup> structural equation.

Curvature as a rotation generator

Cartan's derivation of his two structural equations.

### Cartan's 2<sup>nd</sup> Structural Equation; <sup>35.1</sup>

An explicit representation of the curvature tensor as a (1,1) tensor-valued two-form.

The Riemann curvature tensor is defined by the pointwise linear tensor map

$$R: M_p \times M_p \times M_p \rightarrow M_p$$

$$(W, U, V) \mapsto R(\dots, W, U, V)$$

$$= \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]} W$$

$$= \mathbb{E}_\alpha \otimes w^k \langle \underbrace{dw^l_\alpha + \omega^l_j \wedge \omega^j_\alpha}_{\text{connection form}}, U \wedge V \rangle$$

The explicit representation  $\sum_{k,l} R^l_{kmn} w^m \wedge w^n$

of this tensor is found by letting

$$U = \mathbb{E}_j \langle \omega^j_\alpha, U \rangle$$

$$V = \mathbb{E}_i \langle \omega^i_\alpha, V \rangle$$

$$W = \mathbb{E}_k \langle \omega^k_\alpha, W \rangle \equiv \mathbb{E}_k w^k$$

Just as before, we need to calculate the covariant derivatives

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$$\nabla_V (\mathbb{E}_\alpha w^k) = \nabla_V (\mathbb{E}_\alpha) w^k + \mathbb{E}_\alpha \nabla_V (w^k)$$

$$= \mathbb{E}_\alpha \langle \omega^l_\alpha, V \rangle w^k + \mathbb{E}_\alpha V(w^k)$$

$$\textcircled{1} = \nabla_U \nabla_V (\mathbb{E}_\alpha w^k) = \nabla_U (\mathbb{E}_\alpha \langle \omega^l_\alpha, V \rangle w^k + \mathbb{E}_\alpha V(w^k))$$

$$+ \mathbb{E}_\alpha U \langle \omega^l_\alpha, V \rangle w^k + \mathbb{E}_\alpha U(V(w^k))$$

$$+ \mathbb{E}_\alpha \langle \omega^l_\alpha, V \rangle U(w^k)$$

$$= \mathbb{E}_j \langle \omega^j_\alpha, U \rangle \langle \omega^l_\alpha, V \rangle w^k + \mathbb{E}_\alpha \langle \omega^l_\alpha, U \rangle V(w^k)$$

$$+ \mathbb{E}_\alpha U \langle \omega^l_\alpha, V \rangle w^k + \mathbb{E}_\alpha U(V(w^k))$$

$$+ \mathbb{E}_\alpha \langle \omega^l_\alpha, V \rangle U(w^k)$$

$$\textcircled{2} = \nabla_V \nabla_U (\mathbb{E}_\alpha w^k) = \text{same as above, but with } U \leftrightarrow V.$$

$$\textcircled{3} = \nabla_{[U, V]} (\mathbb{E}_\alpha w^k) = \nabla_{[U, V]} (\mathbb{E}_\alpha) w^k + \mathbb{E}_\alpha \nabla_{[U, V]} w^k$$

$$= \mathbb{E}_\alpha \langle \omega^l_\alpha, [U, V] \rangle w^k + \mathbb{E}_\alpha (UV - VU)(w^k)$$

The expression for the curvature tensor on page 34.13 is

$$\textcircled{1} - \textcircled{2} - \textcircled{3}$$

Aside from the terms linear in  $w^k$ ,  
we have its first derivative terms

$$\begin{aligned} & e_j \langle w^l_a, V \rangle U(w^k) + e_j \langle w^l_a, U \rangle V(w^k) \\ & - e_j \langle w^l_a, U \rangle V(w^k) - e_j \langle w^l_a, V \rangle U(w^k) \\ & = 0, \text{ which cancel, as do the second} \end{aligned}$$

derivative terms

$$e_j U(V(w^k)) - e_j V(U(w^k)) - e_j (UV - VU)(w^k) = 0.$$

Thus we are left with

$$\textcircled{1} - \textcircled{2} - \textcircled{3} =$$

$$\left\{ e_j \langle w^i_p \otimes w^l_a, U \times V \rangle - e_j \langle w^i_p \otimes w^l_a, V \times U \rangle \right. \\ \left. + e_j U(\langle w^l_a, V \rangle) - e_j V(\langle w^l_a, U \rangle) - e_j \langle w^l_a, [U, V] \rangle \right\} w^k$$

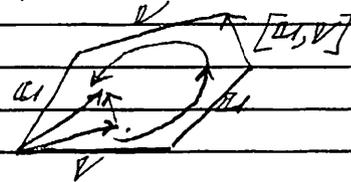
According to the infinitesimal  
version of Stokes' theorem

on page 33,10 we have  $e_j \langle dw^l_a, U \times V \rangle$

Consequently

$$\textcircled{1} - \textcircled{2} - \textcircled{3} = e_j \langle w^i_p \otimes w^l_a - w^l_a \otimes w^i_p + dw^i_p, U \times V \rangle w^k$$

Thus the amount of vectorial change in  $U$ ,  
resulting from its parallel transport  
around the quadrilateral spanned by  $U \times V$



$$= \textcircled{1} - \textcircled{2} - \textcircled{3} = R(\dots, W, U, V) \equiv R(U, V)W$$

$$= e_j \otimes w^k \langle dw^i_p + w^j_l \wedge \omega^l_a, U \times V \rangle$$

$$= e_j \otimes \langle w^k, W \rangle \langle dw^i_p + w^j_l \wedge \omega^l_a, U \times V \rangle$$

$$= e_j \otimes w^k \otimes \Omega^i_p (W, U \times V)$$

Here

$$\Omega^i_p = dw^i_p + w^j_l \wedge \omega^l_a \equiv R^i_{p\,mn} w^m \wedge w^n$$

is the curvature 2-form, which in

$$R = e_j \otimes w^k \otimes \Omega^i_p = e_j \otimes w^k \otimes R^i_{p\,mn} w^m \wedge w^n$$

is the Riemann curvature tensor  $R(\dots, \dots, \dots)$

33.5

MTW's notation is given as

$$\underline{R}^{\mu}_{\nu} = \underline{d}\omega^{\mu}_{\nu} + \omega^{\mu}_{\gamma} \wedge \omega^{\gamma}_{\nu} \quad (= R^{\mu}_{\nu\alpha\beta} \omega^{\alpha} \wedge \omega^{\beta})$$

This is Cartan's second structural equation. It is to be distinguished from Cartan's first structural equation on page 33.4,

$$\underline{\Omega}^{\mu}_{\nu} = \underline{d}\omega^{\mu}_{\nu} + \omega^{\mu}_{\gamma} \wedge \omega^{\gamma}_{\nu} \quad (= T^{\mu}_{\nu\alpha\beta} \omega^{\alpha} \wedge \omega^{\beta})$$

Both equations are 2-forms that characterize the law of parallel transport

$$\underline{d}e_{\nu} = e_{\mu} \otimes \omega^{\mu}_{\nu}$$

which is given by the connection 1-form  $\omega^{\mu}_{\nu}$ .

The two tensors corresponding to the two Cartan structural equations are

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the torsion tensor

$$T = e_{\mu} \otimes \underline{\Omega}^{\mu}_{\nu} = e_{\mu} \otimes (d\omega^{\mu}_{\nu} + \omega^{\mu}_{\gamma} \wedge \omega^{\gamma}_{\nu}),$$

which is a vector valued 2-form

curvature tensor

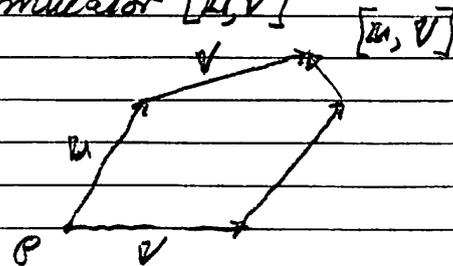
$$R = e_{\mu} \otimes \omega^{\nu} \otimes \underline{\Omega}^{\mu}_{\nu} = e_{\mu} \otimes \omega^{\nu} \otimes (d\omega^{\mu}_{\nu} + \omega^{\mu}_{\gamma} \wedge \omega^{\gamma}_{\nu})$$

which is a (1,1)-tensor valued 2-form.

Both are 2-forms with the implication that both are to be evaluated on an

element of area whose boundary is delimited by vectors  $u, v$ , and their

commutator  $[u, v]$



A) Torsion Induced Displacement. 35,7

Once evaluated on such an element, the vector-valued two form yields a vector,

$$T(u, v) = e_\mu \langle \Omega_\mu^m, u \times v \rangle,$$

which is an infinitesimal displacement

B) Curvature Induced Rotations.

By contrast, evaluation of the (1)-tensor-valued curvature 2-form yields

$$R(\dots, \dots, u, v) = e_\mu \otimes \omega^\nu \langle \Omega_\mu^m, u \times v \rangle$$

which is an infinitesimal rotation.

Indeed, upon parallel transporting the vector

$$W = e_\nu w^\nu$$

around an infinitesimal loop determined

by  $u$  and  $v$ , one finds that this vector points

into a new direction as given by the

$$\text{new vector } W_{\text{NEW}} = e_\mu w_{\text{NEW}}^\mu$$

$$W + R(\dots, W, u, v) = e_\mu w_{\text{old}}^\nu (\delta_\nu^\mu + \langle \Omega_\mu^m, u \times v \rangle)$$

$$= e_\mu w_{\text{old}}^\nu (\delta_\nu^\mu + R^m_{\nu\alpha\beta} u^\alpha v^\beta)$$

$$\equiv e_\mu w_{\text{old}}^\mu$$

Thus the curvature on the area spanned by

$u$  and  $v$  manifests itself as the linear

transformation

$$[\delta_\nu^\mu + R^m_{\nu\alpha\beta} u^\alpha v^\beta] w_{\text{old}}^\nu = w_{\text{NEW}}^\mu$$

The infinitesimal matrix

$$R^m_{\nu\alpha\beta} u^\alpha v^\beta = \langle \Omega_\nu^m, u \times v \rangle$$

is called the generator of that linear

transformation, which differs only

infinitesimally from the identity.

Comment: We shall see that whenever

the parallel transport is induced from a given

metric tensor field then that linear transformation,

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preserves the lengths of vectors, which is  
to say that it's a rotation.

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## Cartan's Method of Deriving his Two Structural Equations

Step 1: Introduce  $dP = e_\mu \otimes \omega^\mu$ , a vector valued diff'l 1-form by the following process of abstraction applied to a Taylor series:

$$\begin{aligned}
 a) \quad f(P+\Delta P) - f(P) &= \underset{rep}{f(x^\mu + \Delta x^\mu)} - \underset{rep}{f(x^\mu)} \\
 &= \Delta x^\mu \frac{\partial f}{\partial x^\mu} + \dots \text{drop the "rep" and} \\
 &= \Delta \tau U^\mu \frac{\partial f}{\partial x^\mu} \quad \text{neglect higher order terms} \\
 &= D_{\Delta \tau U}(f) \quad \left. \begin{array}{l} \text{"change in ppty } f \\ \text{due to displacement} \\ \Delta P = \Delta \tau U: \{ \Delta x^\mu = \Delta \tau U^\mu \} \end{array} \right\} \\
 &= \nabla_{\Delta \tau U}(f) \\
 &= \Delta \tau U(f)
 \end{aligned}$$

$$b) \quad \Delta P = \Delta \tau U: \{ \Delta x^\mu = \Delta \tau U^\mu \}$$

= "change in an as-yet-unspecified ppty due to displacement  $\Delta \tau U$ ."

$$= e_\mu \langle \omega^\mu, \Delta \tau U \rangle \quad \text{expansion relative to some chosen basis}$$

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Hence introduce

$$dP = e_\mu \otimes \omega^\mu = \text{"change in an as-yet-unspecified ppty due to an as-yet-unspecified displacement."}$$

("the infinitesimal")  
Cartan calls this simply "displacement vector."

However, post WWII mathematicians (like Claude Chevalley) observed that

$$dP(V) = e_\mu \langle \omega^\mu, V \rangle = e_\mu V^\mu = V$$

Thus

$$dP = e_\mu \otimes \omega^\mu$$

is a vector valued differential 1-form, a tensor of rank (1). For valid reasons it is also called Cartan's unit tensor.

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Step 2. Apply the exterior diff'l derivative to  $d\varrho$  and obtain Cartan's torsion tensor:

$$\begin{aligned} d d\varrho &= d(e_\mu \otimes \omega^\mu) \\ &= e_\mu d\omega^\mu + de_\mu \wedge \omega^\mu \\ &= e_\mu d\omega^\mu + e_\nu \omega^\nu \wedge \omega^\mu \quad (\text{Law of Export}) \\ &= e_\mu (d\omega^\mu + \omega^\nu \wedge \omega^\mu) = e_\mu \otimes \Omega^\mu \end{aligned}$$

This is Cartan's 1<sup>st</sup> structural equation, a vector-valued 2-form

Step 3. Apply the exterior derivative to the law of parallel transport,

$d e_\mu = e_\nu \otimes \omega^\nu{}_\mu$ , and obtain the curvature tensor:

$$\begin{aligned} d d e_\mu &= d(e_\nu \otimes \omega^\nu{}_\mu) \\ &= e_\nu d\omega^\nu{}_\mu + de_\nu \wedge \omega^\nu{}_\mu \\ &= e_\nu d\omega^\nu{}_\mu + e_\gamma \omega^\gamma \wedge \omega^\nu{}_\mu \\ &= e_\nu (d\omega^\nu{}_\mu + \omega^\gamma \wedge \omega^\nu{}_\mu) = e_\nu \Omega^\nu{}_\mu \end{aligned}$$

35.13

This is Cartan's 2<sup>nd</sup> structural equation, a (1) tensor-valued 2-form.

Comment:

Note that the tensor product sign  $\otimes$  in step 2 and in step 3 is totally unnecessary, provided the multiplicative vector coefficients of the vector-valued differential forms always remain on the left of differential forms proper.