LECTURE 36

1. Cartan's 1st and 2nd Structural Equation via Cartan's exterior calculus.

[MTW § 14.5]
Cantam's Method of Deriving his two Structural Equation.

Cantam introduced $dp = e \otimes dq$, a vector-valued differential 1-form.

Comment: In spite of its mathematical simplicity, there is a highly sophisticated combination of calculus and linear algebra that led to this.

(1) rank tensor

It involves the process of abstraction from an abstraction in the hierarchy of concepts not once, but twice, one on top of the other.

The principal linear part which gives the dominant contribution and which can be expressed in terms of a number of different notations.

Taylor series expansion whose first term is defined as a derivation. Within the context of a Taylor series the concept of a derivation as a vector can by summarized as follows:

1) For any $C^\infty$ function one has for the difference at two nearby point $\bar{r}$ and $P + \Delta P$

$$f(P + \Delta P) - f(P) = \int_{\Delta P} \{ f_r(x^r + \Delta x^r) - \frac{\partial f}{\partial x^r}(x) \}$$

Dropping the $\{ \}$ one has the following

In Lecture 22 where a vector was
Comment: The function \( f \circ \varphi' \) refers to the representative of \( f \) relative to a particular coordinate system, say \( \varphi(x) = (x^i(x^j))_j \). The \( \varphi \)-coordinate representative of \( f \), \( f \circ \varphi'(x^i) = f_{\varphi}(x^i) \), was introduced and used in on pages 22, 22, 22, 22, 22.
\[ f(x+\delta x) - f(x) = f'(x) \delta x + \frac{1}{2} f''(x) (\delta x)^2 + \text{etc.} \]

higher order terms to be neglected

= \Delta \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{u}} + \ldots

= D_{\mathbf{u}u} (f) + \ldots

= \nabla_{\mathbf{u}} f \cdot \Delta \mathbf{u} + \ldots

\]

The difference \( f(x+\delta x) - f(x) \) as expressed in terms of its dominant linear part, namely

\[ f(x+\delta x) - f(x) = \Delta \mathbf{u} \cdot \nabla_{\mathbf{u}} f \]

holds for all \( C^2 \) functions. This fact allows us to form a new and more abstract concept. One does this by omitting explicit reference to these \( C^2 \) functions (which

in physics are measurable properties) with the understanding that these functions exist but are not specified in expressing the difference.

We indicate this kind of an omission by writing the formula for the difference as

\[ \Delta \mathbf{P} = \Delta \mathbf{Z} \cdot \mathbf{u} \]

"change in an as-yet-unspecified property (such as \( f \)) due to displacement \( \Delta \mathbf{Z} \) \( \delta \mathbf{u} \)

\[ = \left< \mathbf{u}^{\circ} \mid \Delta \mathbf{Z} \cdot \mathbf{u} \right> = \left< \mathbf{u}^{\circ} \mid \delta \mathbf{u} \right> = \left< \mathbf{u}^{\circ} \mid \Delta \mathbf{u} \right> \]
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The boxed expression for $\omega$ is the first

new concept.

One now proceeds towards a still higher

level of abstraction by introducing

the dual basis of coordinate $\gamma$ for $M^p$:

\[ \omega^\gamma(e_\alpha) = \langle \omega_{\gamma\alpha} e_\alpha \rangle = \delta^\gamma_{\alpha} \]

One now has

\[ \Delta \bar{P} = e_{\gamma}(\omega^\gamma\Delta \bar{\gamma}) \]

Observe that for any given $\gamma \in M^p$

This allows us to introduce the linear

map (relation) which is a still more

abstract concept, namely

\[ \Delta \gamma \mapsto \Delta \bar{P}(\Delta \gamma) = \Delta \bar{P} = e_{\gamma}(\omega^\gamma(\Delta \bar{\gamma})) \]

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Thus, by again omitting explicit reference

but this time to $\Delta \gamma$, one obtains

\[ \Delta \bar{P} = e_{\gamma}(\omega^\gamma) = \text{"change in an as-yet-}

an as-yet-unspecified displacement"

unspecified displacement".

Cartan calls this simply the "infinitesimal displacement vector". However, pre-World War II mathematicians (like Claude Chevalley) observed that

\[ \Delta \bar{P}(v) = e_{\gamma}(\omega^\gamma, v) = e_{\gamma} v \]

which in retrospect is not rocket science, but very important nevertheless.

Thus, \( \Delta \bar{P} = e_{\gamma}(\omega^\gamma) \)

is a vector-valued differential 1-form, a tensor of rank (1). For valid reasons

it is also called Cartan’s unit tensor.

With this conceptual foundation, Cartan proceeded as follows:
Step 1: Apply the exterior derivative to $\omega^\mu$ and obtain Cartan's torsion tensor:

$$d\omega^\mu = d(e_\nu \wedge \omega^\mu)$$

$$= e_\nu \wedge \omega^\mu + e_\nu \wedge \omega^\mu \wedge \omega^\rho$$

$$= e_\nu \wedge (\omega^\rho + \omega^\rho \wedge \omega^\nu) = e_\nu \wedge (\omega^\rho + \omega^\rho \wedge \omega^\nu)$$

This is Cartan's 1st structure equation, a vector-valued 2-form.

Step 2: Apply the exterior derivative to the law of parallel transport:

$$d\omega^\mu = d(e_\nu \wedge \omega^\mu)$$

$$= e_\nu \wedge \omega^\mu + e_\nu \wedge \omega^\mu \wedge \omega^\rho$$

$$= e_\nu \wedge (\omega^\rho + \omega^\rho \wedge \omega^\nu)$$

This is Cartan's 2nd structural equation, a (1) tensor-valued 2-form.

Comment:

Note that the tensor product sign $\wedge$ in Step 2 and in Step 3 is totally unnecessary, provided the multiplicative coefficients of the vector-valued differential forms always remain on the left of differential forms proper.