

## LECTURE 38

I. The Fundamental Equations of Differential  
Geometry

II. Metric-induced symmetries of

the curvature tensor

[MTW § 11.6, 14.5, Exercise 14.9]

## The Fundamental Equations of Differential Geometry

The mathematical essence of differential geometry consists of a constellation of four fundamental equations.

| GO TO P38.3a |

The first two are based on the phenomenon of parallel transport. The second two, which are based on the measurement of distance, lengths, and angles, relate these measurements to the parallel transport phenomenon in the form of a compatibility condition and its consequence.

The identification of this mathematical essence is the culmination of more than two and a half thousand years of mathematical thinking and we owe its statement to the inductive mathematical genius Elie Cartan.

His original statement was in terms of differentials, i.e. differential forms. An alternative but equivalent statement, developed after WW II, is in terms of operators, namely directional derivatives.

The relation between a differential and a directional derivative is very close:

the former is a slightly more abstract

concept than the latter. (A very succinct, precise and appealing discussion can be found in "Mathematical Analysis" by Tom Apostol, (1957), ch. 6.)

The two alternate ways of exhibiting that constellation of those four fundamental equations is as follows. INSERT

GIVEN: Consider a law of parallel transport  
 $\nabla(v) = \nabla(e_i; v^i)$

$$\boxed{\begin{array}{l} a) \nabla_{e_k} e_i = e_j \Gamma_{ik}^j \quad (= \langle d e_i, e_k \rangle = e_j \langle \omega^j, e_k \rangle) \\ b) \nabla e_i = d e_i = e_j \omega^j{}_i \end{array}}$$

It is characterized by the two structural equations for torsion & curvature:

I.)

$$1. a) T(u, v) = \nabla_u v - \nabla_v u - [u, v] = e_i T^i_{mn} u^m v^n$$

$$b) e_i (d\omega^i + \omega^i{}_j \wedge \omega^j) = e_i T^i_{jmn} \omega^m \omega^n$$

$R(u, v)w$

38.4

II.

2. a)

$$R(W, U, V) = (\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]}) W = e_i \cdot R^i_{jlmn} u^m v^n w^j$$

$$b) d\omega^i_j + \omega^i_m \omega^l_m \omega^e_j = R^i_{jlmn} \omega^m \omega^n$$

and the metric compatibility condition

III.

3. a)

$$v = e_i \\ w = e_j$$

$$\nabla_u(v \cdot w) = (\nabla_u v) \cdot w + v \cdot (\nabla_u w)$$

$$b) \nabla_u g_{ij} = \omega^e_i g_{ej} + g_{ie} \omega^e_j = \omega^e_{j|i} + \omega^e_{i|j}$$

together with its consequence [GO TO P 38.5]

IV

4.

$$\rightarrow a) \circ = R(u, v)(w, x) = [R(u, v)w] \cdot x + w \cdot [R(u, v)x]$$

$$= (g_{ei} R^i_{jlmn} x^l w^j + g_{ei} R^i_{jlmn} w^e x^j) u^m v^n$$

$$= (R_{ej'mn} + R_{jemn}) x^l w^j u^m v^n$$

$\forall u, v, w, x$

INSERT from P 38.4C

b) Solution to Problem 1 of Homework #8

$$\rightarrow \circ = d(e_i \cdot e_j) = e_j \cdot e_k S^k_{imj} + e_i \cdot e_k S^k_{imj}$$

$$\rightarrow \text{where } S^k_{imj} \equiv \omega^k_{i|m} + \omega^k_{2|1} \omega^l_{m|i} = R^k_{imjn} \omega^m \omega^n$$

38.5

[u, v]

$\rho_{4\Delta P}$

u

$\rho$

v

$\rho + SP$

u

w

x

w

isograms of W-X

Consider

- a) the infinitesimal quadrilateral spanned by the vector fields  $W$  and  $X$
- b) the scalar field  $W \cdot X$  which is formed from the inner product of the two vector fields  $W$  and  $X$ .
- c) Parallel transporting  $W$  and  $X$  around the boundary of the quadrilateral and requiring the transport to be metric compatible

$$P \rightarrow P + \Delta P;$$

$$W_{P+\Delta P} \cdot X_{P+\Delta P} - W_P \cdot X_P = \Delta \tau \nabla_u (W \cdot X)$$

$$= \underbrace{\left( \begin{array}{l} \text{vector at} \\ P+\Delta P \text{ (to} + \Delta \tau \nabla_u W \text{)} \end{array} \right) \cdot \left( \begin{array}{l} \text{vector at} \\ P+\Delta P \text{ (to} + \Delta \tau \nabla_u X \text{)} \end{array} \right)}_{W_P \cdot X_P} - W_P \cdot X_P =$$

L

|| + [metric compatibility condition]

$$W_P \cdot X_P$$

$$= \Delta \tau (\nabla_u W \cdot X + W \cdot \nabla_u X) = \nabla_u (W \cdot X)$$

38.7

For  $\rho \rightarrow \rho + \delta\rho$  we have similarly

$$(\nabla_{\rho} w) \cdot x + w \cdot \nabla_{\rho} x = \nabla_{\rho}(w \cdot x)$$

38.8

$$\begin{aligned} & \left[ \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \right] f \\ &= \cancel{\nabla_u \nabla_v(f)} - \cancel{\nabla_v \nabla_u(f)} - (u \nabla_v(f) - v \nabla_u(f)) = 0 \end{aligned}$$

For a) we have  
the metric-induced constraint on curvature,

38.9

$$R_{\ell jmn} = -R_{j\ell mn}$$

is valid because of the metric compatibility condition 3.a). Indeed, observe that

$$w \cdot X = f$$

is a scalar function. Consequently,

$$\begin{aligned} (\nabla_u \nabla_v - \nabla_v \nabla_u) f &= (\partial_u \partial_v - \partial_v \partial_u) f \\ &= [u, v] f \end{aligned}$$

It follows that

$$\begin{aligned} R(u, v)(w \cdot X) &= (\nabla_u \nabla_v - \nabla_v \nabla_u - [u, v]) f = \\ &= 0 \end{aligned} \tag{1}$$

On the other hand, using the metric compatibility condition, one has

$$0 = [\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[uv]}] W \cdot X = \\ = (\nabla_u \nabla_v W) \cdot X - (\nabla_v \nabla_u W) \cdot X + W (\nabla_u \nabla_v X - \nabla_v \nabla_u X) \\ - [\nabla_{[uv]} W] \cdot X - W (\nabla_{[uv]} X)$$

$$\text{Let } X = e_i$$

$$W = e_j$$

$$U = e_m$$

$$V = e_n$$

$$0 = \underbrace{e_i R^{\ell}_{jmn} \cdot e_i}_{{g_{\ell i}}} + \underbrace{e_j \cdot e_\ell R^{\ell}_{imn}}_{{g_{j\ell}}}$$

$$0 = R_{ijmn} + R_{jimn}$$

Hence the metric induced by  $\pi$   
implies

$$\boxed{R_{ijmn} = -R_{jimn}}$$