

LECTURE 40

Metric on S^3

Metric for a Robertson-Walker universe

Curvature inside a spherical star in

free fall collapse;

its Einstein tensor [MTW Box 14.5]

Let us consider a 3-sphere of radius a in a 4-dimensional Euclidean space spanned by (x^1, x^2, x^3, x^4)

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2 : x^1 = a \sin \chi \sin \theta \cos \phi$$

$$\theta = \frac{\pi}{2}$$

$$x^2 = a \sin \chi \sin \theta \sin \phi$$

$$x^3 = a \sin \chi \cos \theta$$

$$x^4 = a \cos \chi$$

The metric tensor for Euclidean space E^4 is

$$\begin{aligned} (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 &= \\ &= da^2 + a^2 [dx^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \\ &\equiv da^2 + d\sigma^2 \end{aligned}$$

We now consider the spacetime metric

$$ds^2 = -dt^2 + a^2(t) [dx^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]$$

One can show that $d\sigma^2$ expresses the geometry of an isotropic translation invariant space. Its intrinsic geometry is that of the surface of S^3 of radius $a = a(t)$, which we allow to depend on time.

40.1

Curvature of Star in Free Fall Collapse

Ch. 14.1
14.5
14.6

$$ds^2 = -dt^2 + \dot{a}^2(t) [dx^2 + \sin^2 x (d\theta^2 + \sin^2 \theta d\phi^2)]$$

$$= -(\hat{\omega}^t)^2 + (\hat{\omega}^x)^2 + (\hat{\omega}^\theta)^2 + (\hat{\omega}^\phi)^2$$

11.1 13.1
11.2 13.2
11.3 13.3
11.4

A. Step (1): Basis forms dual to ortho normal vectors

$\hat{\omega}^t = dt$
$\hat{\omega}^x = a dx$
$\hat{\omega}^\theta = a \sin x d\theta$
$\hat{\omega}^\phi = a \sin x \sin \theta d\phi$

$$d\omega^u + \omega^u_{\nu} \wedge \omega^\nu = 0$$

B. Step (2): Compute Rotation coefficients

$$(1) d\hat{\omega}^t = 0$$

$$\frac{1}{a} \hat{\omega}^x = \dot{a} dt \wedge dx = \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^x$$

$$d\hat{\omega}^\theta = \dot{a} \sin x dt \wedge d\theta + a \cos x dx \wedge d\theta$$

$$= \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^\theta + \frac{1}{a} \frac{\cos x}{\sin x} \hat{\omega}^x \wedge \hat{\omega}^\theta$$

$$d\hat{\omega}^\phi = \dot{a} \sin x \sin \theta dt \wedge d\phi + a \cos x \sin \theta dx \wedge d\phi$$

$$+ a \sin x \cos \theta d\theta \wedge d\phi$$

$$= \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^\phi + \frac{\cos x}{a \sin x} \hat{\omega}^x \wedge \hat{\omega}^\phi + \frac{\cos \theta}{a \sin \theta \sin x} \hat{\omega}^\theta \wedge \hat{\omega}^\phi$$

(2) Equate to $\hat{\omega}^t \wedge \hat{\omega}^\theta$ (i.e. to $-\hat{\omega}^\theta \wedge \hat{\omega}^x$)

$$(a) d\hat{\omega}^t = \hat{\omega}^t_{\Lambda} \hat{\omega}^t + \hat{\omega}^x_{\Lambda} \hat{\omega}^x + \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta + \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi$$

$$(2) \hat{\omega}^t_{\Lambda} = 0 \quad \hat{\omega}^t_{\Lambda} = -\hat{\omega}^x_{\Lambda} \quad \text{why is antisymmetric}$$

$$(11) \hat{\omega}^x_{\Lambda} \hat{\omega}^x = 0 \Rightarrow \hat{\omega}^x_{\Lambda} = (\dots) \hat{\omega}^x$$

$$(11) \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta = 0 \Rightarrow \hat{\omega}^\theta_{\Lambda} = (\dots) \hat{\omega}^\theta \quad \text{mod C 3a}$$

$$(11) \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi = 0 \Rightarrow \hat{\omega}^\phi_{\Lambda} = (\dots) \hat{\omega}^\phi$$

(b) Similarly

$$d\omega^x = \hat{\omega}^t_{\Lambda} \hat{\omega}^x + \hat{\omega}^x_{\Lambda} \hat{\omega}^x + \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta + \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi$$

$$(1) \hat{\omega}^t_{\Lambda} \hat{\omega}^x = \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^x \Rightarrow \hat{\omega}^x_{\Lambda} = \frac{\dot{a}}{a} \hat{\omega}^x + (\dots) \hat{\omega}^t \quad \text{zero from 2a(11)}$$

$$(2) \hat{\omega}^\theta_{\Lambda} \hat{\omega}^x = \hat{\omega}^\theta_x = -\hat{\omega}_{xx} = 0; \quad \hat{\omega}^x_{\Lambda} = -\hat{\omega}_x \hat{\omega}^x = \hat{\omega}^x_{\Lambda}$$

$$(3) \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta = 0 \Rightarrow \hat{\omega}^\theta_{\Lambda} = (\dots) \hat{\omega}^\theta \leftarrow \text{C 3a}$$

$$(4) \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi = 0 \Rightarrow \hat{\omega}^\phi_{\Lambda} = (\dots) \hat{\omega}^\phi \leftarrow \text{C 3a}$$

$$(c) d\omega^\theta = \hat{\omega}^t_{\Lambda} \hat{\omega}^\theta + \hat{\omega}^x_{\Lambda} \hat{\omega}^\theta + \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta + \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\theta$$

$$(1) \hat{\omega}^t_{\Lambda} \hat{\omega}^\theta = \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^\theta$$

$$\Rightarrow \hat{\omega}^\theta_{\Lambda} = \frac{\dot{a}}{a} \hat{\omega}^\theta + (\dots) \hat{\omega}^t$$

$$(2) \hat{\omega}^x_{\Lambda} \hat{\omega}^\theta = \frac{1}{a} \frac{\cos x}{\sin x} \hat{\omega}^x \wedge \hat{\omega}^\theta \quad \text{zero from 2a(11)}$$

$$\Rightarrow \hat{\omega}^\theta_{\Lambda} = \frac{1}{a} \frac{\cos x}{\sin x} \hat{\omega}^\theta + (\dots) \hat{\omega}^x$$

$$(3) \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\theta = \hat{\omega}_{\theta\theta} = 0$$

$$(4) \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\theta = 0 \Rightarrow \hat{\omega}^\phi_{\Lambda} = (\dots) \hat{\omega}^\phi$$

$$(d) d\omega^\phi = \hat{\omega}^t_{\Lambda} \hat{\omega}^\phi + \hat{\omega}^x_{\Lambda} \hat{\omega}^\phi + \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\phi + \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi$$

$$(1) \hat{\omega}^t_{\Lambda} \hat{\omega}^\phi = \frac{\dot{a}}{a} \hat{\omega}^t \wedge \hat{\omega}^\phi \Rightarrow \hat{\omega}^\phi_{\Lambda} = \frac{\dot{a}}{a} \hat{\omega}^\phi + (\dots) \hat{\omega}^t$$

$$(2) \hat{\omega}^x_{\Lambda} \hat{\omega}^\phi = \frac{\cos x}{a \sin x} \hat{\omega}^x \wedge \hat{\omega}^\phi \Rightarrow \hat{\omega}^\phi_{\Lambda} = \frac{\cos x}{a \sin x} \hat{\omega}^\phi + (\dots) \hat{\omega}^x \quad \text{zero from 2b(11)}$$

$$(3) \hat{\omega}^\theta_{\Lambda} \hat{\omega}^\phi = \frac{\cos \theta}{a \sin \theta \sin x} \hat{\omega}^\theta \wedge \hat{\omega}^\phi \Rightarrow \hat{\omega}^\phi_{\Lambda} = \frac{\cos \theta}{a \sin \theta \sin x} \hat{\omega}^\phi + (\dots) \hat{\omega}^\theta \quad \text{zero from 2b(11)}$$

$$(4) \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi = \frac{\cos \phi}{a \sin \phi \sin x} \hat{\omega}^\phi \wedge \hat{\omega}^\phi \Rightarrow \hat{\omega}^\phi_{\Lambda} = \frac{\cos \phi}{a \sin \phi \sin x} \hat{\omega}^\phi + (\dots) \hat{\omega}^\phi \quad \text{zero from 2c(11)}$$

$$(5) \hat{\omega}^\phi_{\Lambda} \hat{\omega}^\phi = \hat{\omega}_{\phi\phi} = 0$$

40,3

The results are therefore:

$$\omega_z^t = 0; \omega_x^t = \frac{\dot{a}}{a} \hat{\omega}^x; \omega_y^t = \frac{\dot{a}}{a} \omega^y; \omega_\phi^t = \frac{\dot{a}}{a} \omega^\phi$$

$$= \omega_x^t = \omega_y^t = \omega_\phi^t$$

$$\omega_x^t = 0; \omega_y^t = \frac{1}{a} \frac{\cos X}{\sin X} \omega^\phi; \omega_\phi^t = \frac{\cos X}{\sin X} \omega^\phi = -\omega_\phi^t$$

$$\omega_\theta^t = 0; \omega_\phi^t = \frac{\cos \theta}{\sin \theta} \frac{1}{a} \omega^\phi = -\omega_\theta^t$$

$$\omega_\phi^t = 0$$

C Step 3 Calculate curvilinear form

$$\hat{\Sigma}_\mu^t = dw_\mu^t + \omega_\gamma^t w_\lambda^\gamma$$

$$a) \hat{\Sigma}_x^t = dw_x^t + \omega_y^t w_x^\gamma = dw_x^t + \underbrace{\omega_y^t}_{\text{zero}} \underbrace{w_x^\gamma}_{\text{zero}}$$

$$= d(\dot{a} dx)$$

$$= \ddot{a} dt dx$$

$$= \frac{\ddot{a}}{a} w_t^t w_x^t$$

$$b) \hat{\Sigma}_\theta^t = dw_\theta^t + \omega_\phi^t w_\theta^\gamma = dw_\theta^t + \underbrace{\omega_\phi^t}_{\text{zero}} \underbrace{w_\theta^\gamma}_{\text{zero}}$$

$$= d(\dot{a} \sin X d\theta) - \frac{\dot{a}}{a} \frac{\cos X}{\sin X} w_\lambda^x w_\theta^t$$

$$= \frac{\ddot{a}}{a} w_\lambda^x w_\theta^t + \frac{\dot{a}}{a^2} \frac{\cos X}{\sin X} w_\lambda^x w_\theta^t - \frac{\dot{a}}{a^2} \frac{\cos X}{\sin X} w_\lambda^x w_\theta^t$$

$$R_{\theta t}^x$$

3

$$c) \hat{\Sigma}_\phi^t = dw_\phi^t - \omega_y^t w_\lambda^\phi = dw_\phi^t + \underbrace{a^2 \sin X w_\lambda^x w_\phi^t}_{\text{cancel}} + \underbrace{w_\lambda^t w_\phi^t}_{\text{cancel}}$$

$$= d(\dot{a} \omega^\phi) - \frac{\dot{a}}{a^2} \frac{\cos X}{\sin X} w_\lambda^x \dot{a} \omega^\phi - \frac{\dot{a}}{a^2} \frac{\cos X}{\sin X} w_\lambda^x \omega^\phi$$

$$= d(\dot{a} \sin X \sin \theta d\phi)$$

$$= \underbrace{\ddot{a}}{a} w_t^t w_\phi^t + \frac{\dot{a}^2 \cos X}{a^2 \sin X} w_\lambda^x w_\phi^t + \frac{\dot{a}^2 \cos \theta}{a^2 \sin \theta} w_\lambda^x w_\phi^t$$

$$R_{\phi t}^x$$

$$d) \hat{\Sigma}_\theta^x = dw_\theta^x + \omega_y^t w_\lambda^\theta = dw_\theta^x + w_{t1}^x w_\theta^t + \underbrace{w_{\phi 1}^x w_\theta^t}_{\text{zero}}$$

$$= d\left(-\frac{1}{a} \frac{\cos X}{\sin X} \omega^\theta\right) + \frac{\dot{a}^2}{a^2} w_\lambda^x w_\theta^t$$

$$= d(-)(\cos X d\theta) + \frac{\dot{a}^2}{a^2} w_\lambda^x w_\theta^t$$

$$= + \sin X dX d\theta + \text{..}$$

$$= \frac{1}{a^2} w_\lambda^x w_\theta^t + \frac{\dot{a}^2}{a^2} w_\lambda^x w_\theta^t$$

$$= + \left(\frac{1}{a^2} + \frac{\dot{a}^2}{a^2}\right) w_\lambda^x w_\theta^t$$

$$\hat{R}_\theta^x$$

$$e) \hat{\Sigma}_\phi^x = dw_\phi^x + \omega_y^t w_\lambda^\phi = dw_\phi^x + w_{t1}^x w_\phi^t + w_{\theta 1}^x w_\phi^t$$

$$= d\left(-\frac{\cos X}{a \sin X} \omega^\phi\right) + (+) \frac{\dot{a}^2}{a^2} w_\lambda^x w_\phi^t + \frac{1}{a^2} \frac{\cos X \cos \theta}{\sin X \sin \theta} w_\lambda^x w_\phi^t$$

$$= d(-\cos X \sin \theta d\phi) + \frac{\dot{a}^2}{a^2} w_\lambda^x w_\phi^t + \cos X \cos \theta d\theta d\phi$$

$$= + \sin X \sin \theta dX d\phi - \cos X \cos \theta d\theta d\phi + \text{..}$$

$$= \frac{1}{a^2} w_\lambda^x w_\phi^t + \frac{\dot{a}^2}{a^2} w_\lambda^x w_\phi^t = \underbrace{\frac{1+\dot{a}^2}{a^2} w_\lambda^x w_\phi^t}_{\text{cancel}}$$

$$\hat{R}_{\phi x}^x$$

40,4

4

40.5

$$\begin{aligned}
 f) \hat{R}_{\varphi}^{\theta} &= d\omega_{\varphi}^{\theta} + \omega_{\varphi}^{\theta} \omega_{\varphi}^{\delta} = d\omega_{\varphi}^{\theta} + \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta} + \omega_{\varphi}^{\theta} \omega_{\varphi}^{\delta} \\
 &= d\left(\frac{\cos\theta}{\sin\theta \sin\varphi}\right) + \frac{\dot{a}^2}{a^2} \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta} + (-)\frac{1}{\sin\theta \sin\varphi} \cos\theta \cos\varphi \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta} \\
 &= d(-\cos\theta d\varphi) + \frac{\dot{a}^2}{a^2} \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta} - \cos\theta \sin\theta d\theta d\varphi \\
 &= \sin\theta d\theta d\varphi \quad + \downarrow \quad - \quad \downarrow \\
 &= + \sin^2\theta \sin\theta d\theta d\varphi + \downarrow \\
 &= + \frac{1}{a^2} \omega_{\theta}^{\theta} \cos\theta + \frac{\dot{a}^2}{a^2} \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta} \\
 &= + \underbrace{\frac{1+\dot{a}^2}{a^2}}_{\hat{R}^{\theta}_{\varphi\varphi}} \omega_{\theta}^{\theta} \omega_{\varphi}^{\delta}
 \end{aligned}$$

g) Conclusion: Physical components of curvature tensor,

$\hat{R}_{xix}^t = \frac{\ddot{a}}{a}$	$\hat{R}_{\varphi\varphi\varphi}^{\theta} = \frac{1+\dot{a}^2}{a^2}$	all others are obtained by suitable symmetry $R_{\alpha\beta\gamma\delta}^M = -R_{\gamma\delta\alpha\beta}^M$
$\hat{R}_{\theta t\theta}^t = \frac{\ddot{a}}{a}$	$\hat{R}_{\varphi x\varphi}^x = \frac{1+\dot{a}^2}{a^2}$	
$\hat{R}_{\theta t\varphi}^t = \frac{\ddot{a}}{a}$	$\hat{R}_{\theta x\theta}^x = \frac{1+\dot{a}^2}{a^2}$	

$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$

D) Ricci tensor physical components

$R_{\mu\nu} = R_{\mu\alpha\nu\alpha}^{\alpha}$

$\hat{R}_{tt}^t = \hat{R}_x^x + \hat{R}_{\varphi\varphi}^{\theta} + \hat{R}_{\theta\theta}^t + \hat{R}_{\varphi\varphi}^t = -3\frac{\ddot{a}}{a}$

$\hat{R}_{xx}^t = \hat{R}_{xxt}^t + \hat{R}_{\varphi x\varphi}^{\theta} + \hat{R}_{\theta x\theta}^x = \frac{\ddot{a}}{a} + 2\frac{1+\dot{a}^2}{a^2} = \frac{2+2\dot{a}^2}{a^2}$

$\hat{R}_{\theta\theta}^t = \hat{R}_{\theta t\theta}^t + \hat{R}_{\varphi\theta\varphi}^{\theta} + \hat{R}_{\theta\varphi\theta}^{\varphi} = \text{same as } \hat{R}_{xx}^t$

 $R_{\varphi\varphi} = \text{same; others are zero}$

note the minus sign!

E) Curvature invariant

$$\begin{aligned}
 R &= R_{tt}^t + R_{xx}^x + R_{\theta\theta}^{\theta} + R_{\varphi\varphi}^{\varphi} \\
 &= \frac{3\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\frac{1+\dot{a}^2}{a^2}\right) \\
 &= 6 \frac{\ddot{a} + \dot{a}^2 + 1}{a^2}
 \end{aligned}$$

F) Einstein tensor physical components

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$G_{\mu\nu} = 0 \text{ for } \mu \neq \nu$

$$\begin{aligned}
 G_{tt} &= R_{tt} - \frac{1}{2}(-)R = -\frac{3\ddot{a}}{a} + 3\frac{\ddot{a} + \dot{a}^2 + 1}{a^2} \\
 &= 3\frac{\dot{a}^2 + 1}{a^2}
 \end{aligned}$$

$$\begin{aligned}
 G_{xx} &= R_{xx} - \frac{1}{2}R = \frac{\ddot{a}}{a} + 2\frac{1+\dot{a}^2}{a^2} - 3\frac{\ddot{a} + \dot{a}^2 + 1}{a^2} \\
 &= -\frac{2\ddot{a} + \dot{a}^2 + 1}{a^2}
 \end{aligned}$$

$G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2}R = G_{xx}$

$G_{\varphi\varphi} = G_{xx} \dots$

$\left(\frac{\dot{a}^2 + 1}{a^2}\right) = \frac{2\ddot{a}}{a^2} - 2\frac{(\dot{a}^2 + 1)\dot{a}}{a^3} = \frac{\dot{a}}{a} \frac{2\ddot{a} - 2\dot{a}^2 - 2}{a^2} = \dot{a}$