Lagrangian Mechanics and the Three-body Problem

Agenda:

1. What facts of reality give rise to Lagrangian Mechanics?
2. The restricted planar three-body problem.

   a) dynamics relative to a rotating frame via dynamics in a combined magnetic and electric field.
   b) Jacobi's integral of motion.
   c) forbidden regions and the topology of mathematically unquantified regions.
   d) the five libration points of Lagrange.

The theme which unites 1. and 2. is "transformations". They form a key connecting link which allows a mathematician to think like a physicist and a physicist like a mathematician, to the advantage of both. The transformations are to a curvilinear cordinate frame, i.e. to an accelerated frame in 1. and to a rotating frame in 2.

1 Lagrange's Equations of Motion

What is their physical origin?

A. Launch a particle vertically from x_1 at time t_1, watch it reach its maximum height, and then catch it at time t_2 at the instant it is located at x_2.

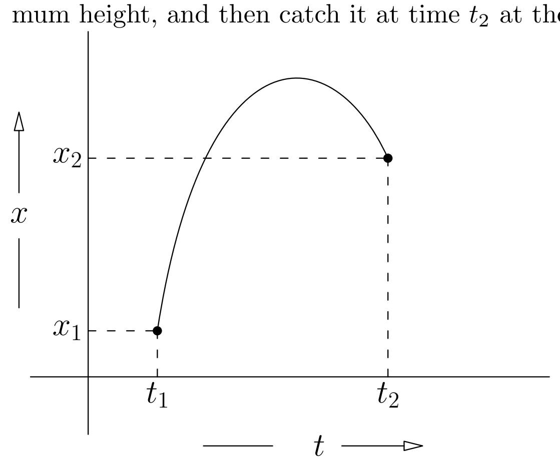


Figure 1: Spacetime trajectory of a particle thrown into the air.

From Galileo we learned that in its travel from (t_1, x_1) to (t_2, x_2) the particle traces a space-time trajectory which is given by a parabola. Why so? Answer:

1. Newton's 1st Law: Every body continues in its state of rest, or of uniform motion in a straight line, unless it iscompelled to change that state by forces impressed upon it.

2. The principle of equivalence.

B. Simpler case: Free Particle.
Consider the motion of a particle moving freely in a free float ("inertial") frame. This particle moves with constant velocity, i.e. its space-time trajectory is a straight line.

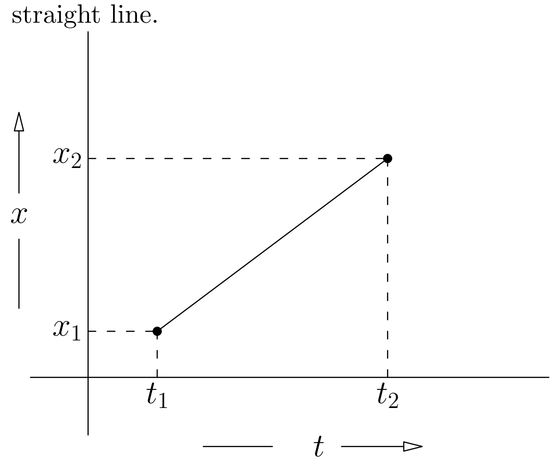


Figure 2: Spacetime trajectory of a free particle is a straight line.

The implication of this fact is that for such a curve the integral

$$\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \left(\frac{dx(t)}{dt} \right)^2 dt \equiv \langle v^2 \rangle = min\,!$$

as compared to other curves having the same starting and termination points.
Q: Why?

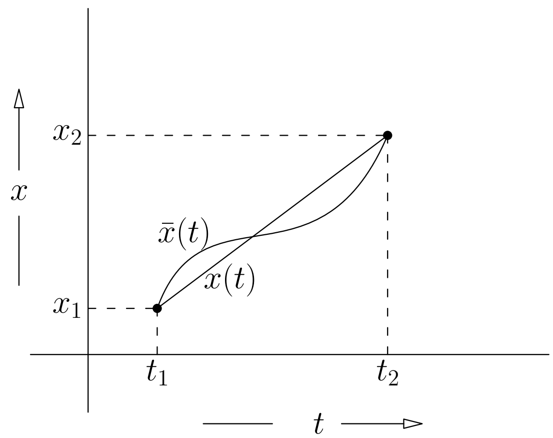


Figure 3: Straight line $x(t)$ and its variant $\bar{x}(t)$ have the same average velocity: $\langle \bar{v} \rangle = v$ $(= const.)$.

A: All such curves have the same end points,

$$\begin{aligned} \bar{x}(t_1) &= x(t_1) \\ \bar{x}(t_2) &= x(t_2). \end{aligned}$$

Thus they all have the same average velocity,

$$\underbrace{\frac{\bar{x}(t_2) - \bar{x}(t_1)}{t_2 - t_1}}_{\langle \bar{v} \rangle} = \underbrace{\frac{x(t_2) - x(t_1)}{t_2 - t_1}}_{\langle v \rangle} .$$

Consequently,

$$\langle \bar{v} \rangle = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \underbrace{\frac{d\bar{x}}{dt}}_{\bar{v}(t)} dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \underbrace{\frac{dx}{dt}}_{v(t) = const \equiv v} dt = \langle v \rangle, \qquad (1)$$

which means that the areas under the curves $\bar{v}(t)$ and $v(t) = v$ are the same.

Applying this fact to the positivity of the averaged squared deviation (away from the average), using the fact that $\bar{v} = \langle \bar{v} \rangle$ and that $\langle \langle \bar{v} \rangle \langle \bar{v} \rangle \rangle = \langle \bar{v} \rangle \langle \bar{v} \rangle$, one finds with the help of Eq.(1) that

$$0 \leqslant \langle (\bar{v} - \langle \bar{v} \rangle)^2 \rangle = \langle \bar{v}^2 \rangle - (\langle \bar{v} \rangle)^2 = \langle \bar{v}^2 \rangle - (\langle v \rangle)^2 .$$

Consequently,

$$\langle \bar{v}^2 \rangle \geqslant (\langle v \rangle)^2 = v^2,$$

or

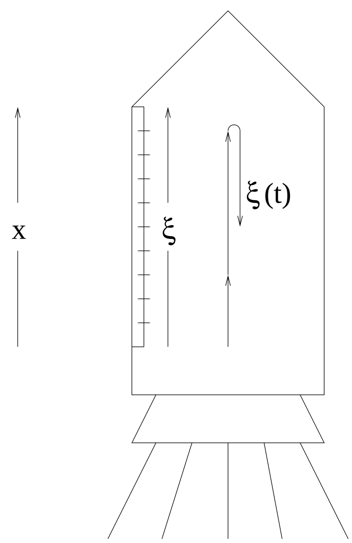
$$\int_{t_i}^{t_2} \underbrace{\left(\frac{d\bar{x}(t)}{dt}\right)^2}_{\substack{\text{for } \textbf{any} \\ \text{non-straight} \\ \text{line}}} dt \geqslant \int_{t_i}^{t_2} \left(\frac{dx(t)}{dt}\right)^2 dt.$$

This says that a free particle (one whose spacetime trajectory is a straight line) moves so that the integral of its kinetic energy is a minimum:

$$\int_{t_i}^{t_2} K.E.\, dt \equiv \int_{t_i}^{t_2} \frac{1}{2} m \left(\frac{dx(t)}{dt}\right)^2 dt = min\,!$$

C. Free particle in an accelerated frame.

Consider the motion of the same particle moving freely in a frame accelerated uniformly with acceleration g.



A point ξ fixed in the accelerated frame will move relative to the free float frame according to

$$x = \xi + \frac{1}{2} g t^2.$$

It follows that, relative to the accelerated frame, the spacetime trajectory of the particle, $\xi(t)$, is given by

$$x(t) = \xi(t) + \frac{1}{2} g t^2. \tag{2}$$

Here $x(t)$ is the linear trajectory in Figure 2.

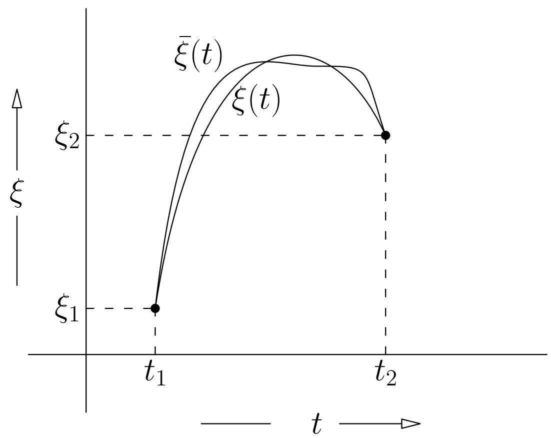


Figure 4: Minimizing trajectory $\xi(t)$ and one of its variants $\bar{\xi}(t)$.

The to-be-minimized integral takes the form

$$
\begin{aligned}
min = \int_{t_i}^{t_2} \left(\frac{dx(t)}{dt}\right)^2 dt &= \int_{t_i}^{t_2} \left(\frac{d\xi}{dt} + gt\right)^2 dt \\
&= \int_{t_i}^{t_2} \left\{ \left(\frac{d\xi}{dt}\right)^2 + 2gt\frac{d\xi}{dt} + g^2t^2 \right\} dt \\
&= \int_{t_i}^{t_2} \left\{ \left(\frac{d\xi}{dt}\right)^2 - 2g\xi \right\} dt + 2gt\xi|_{t_1}^{t_2} + \frac{1}{3}gt^2|_{t_1}^{t_2}
\end{aligned}
$$

The last line is the result of an integration by parts. The last two terms are the same for all trajectories passing through the given points (t_1, x_1) and (t_2, x_2). Consequently,

$$
\int_{t_i}^{t_2} \frac{1}{2} m \left(\frac{dx(t)}{dt}\right)^2 dt = min \iff \int_{t_i}^{t_2} \left\{ \frac{m}{2} \left(\frac{d\xi}{dt}\right)^2 - mg\xi \right\} dt = min
$$

D. Free particle in an *equivalent* gravitational field.

The *equivalence principle* is an observation of the fact that in an accelerated frame the laws of moving bodies are the same as those in a homogeneous gravitational field.

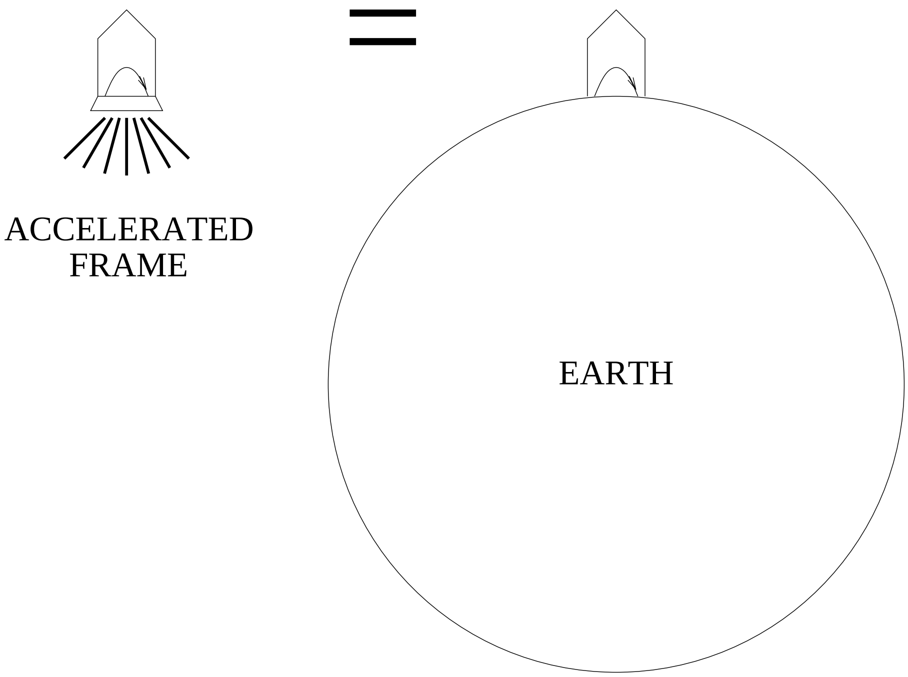


Figure 5: Trajectories in an accelerated frame are indistinguishable from those in a gravitational field. In particular the motion of particles of different composition (gold, aluminum, snakewood, etc.) is independent of their composition.

Recall that in a gravitational field

$$mg\xi = P.E.$$

represents the potential energy of a mass m at a height ξ. Consequently, the trajectory of a particle in a gravitational field is determined by

$$\int_{t_i}^{t_2} (\,K.E. - P.E.\,)\,dt \equiv \int_{t_i}^{t_2} L(\dot{x},\,x,\,t)\,dt = min.$$

In fact, the trajectory of a particle which satisfies this minimum condition satisfies the Euler-Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

which is Newton's second law of motion

$$ma = F$$

for the one-dimensional motion of a particle.
Nota bene:

1. The same minimum priciple hold even if g, and hence the potential energy $P.E.$, depends explicitly on time.
2. This principle is a special case of what is known as Hamilton's principle of least action. The difference is that the latter also accomodates motion which are subject to constraints.

E. Extension to multi dimensions and generic potentials.

The Lagrangian formulation opens new vistas on the notion of bodies. It can be fruitfully implemented for more general motions and potentials. These generalizations are alternate but equivalent formulations of Newtonian mechanics. They are simply expressed by the statement that

$$\int_{t_i}^{t_2} (\, K.E. - P.E.\,)\, dt = min$$

with

$$K.E. = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\vec{x}}_i \cdot \dot{\vec{x}}_i$$

$$P.E. = U(t,\, \vec{x_i})$$

on the class of all system trajectories having fixed endpoints.

The advantage of Lagrangian Mechanics becomes evident in the process of setting up Newton's equations of motion. In Newtonian Mechanics one must do this for each force component separately, a task which becomes non-trivial relative to curvilinear coordinate frames (spherical, cylindrical, etc.). By contrast, in the Lagrangian approach one merely identifies the two scalars $K.E$ and $P.E.$ relative to the coordinate frame of one's choice. The remaining task of setting up the differential equations of motion is done automatically by merely writing down the Euler-Lagrange equations.

2 The Three-body Problem

Taking advantage of the road paved by Newton and Euler, Lagrange asked the following question: Does there exist a configuration of gravitationally interacting bodies which, when launched with appropriate velocities, will execute the motion of three rigidly connected points?

He demonstrated that the answer is “yes”. Considered three masses, M_1, M_2, and M_3, configured into an equilateral triangle with equal sides a.

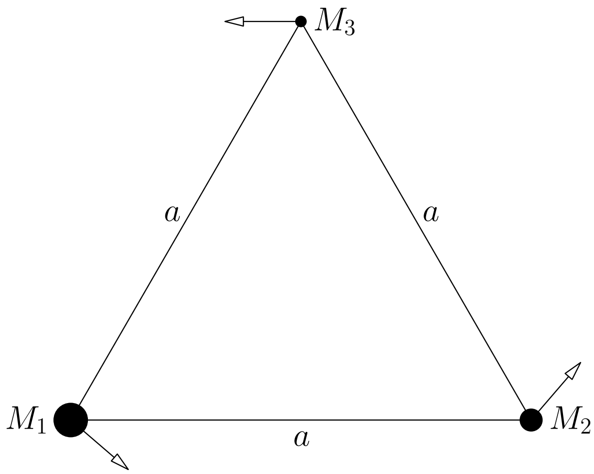


Figure 6: Equilateral planar three-body system in a state of rigid rotation around its center of mass.

If these masses are launched so that their angular velocity around their center of mass satisfies

$$\omega^2 = \frac{G(M_1 + M_2 + M_3)}{a^3} \qquad (\text{"Keppler's third law"})$$

then they will continue to rotate uniformly about their center of mass as if they form a rigid triangle. In other words, in the co-rotating frame the three masses are in a state of equilibrium: Newton’s laws permit a perfect balance between the attractive gravitational force and the repulsive centrifugal force.

The second question is: Is this equilibrium stable or unstable? The answer is given by the inequality

$$(M_1 + M_2 + M_3)^2 > 27(M_1M_2 + M_2M_3 + M_3M_1). \tag{3}$$

If the three masses satisfy this inequality then they form a stable[1]configuration: the triangle will oscillate by changing its area and/or its shape, but it will

[1]To be precise, the equilibrium is *linearly* stable. This means that non-linear perturbations have been ignored. As far as I know, whether it remains stable when one does not ignore these nonlinearities, is a nontrivial open question. However, in the case of the *restricted* three-body problem, where $M_3 \ll M_1, M_2$, so that the gravitational influence of $M_3$is negligible, one can give criteria for *absolute* stability. They are found near the end of this section. Astronomically, the difference between linear and absolute stability is a question of time. The former refers to stability at least in the intermediate future (many orbital revolutions/librations), the latter refers to the whole future.

not disintegrate. The configuration behaves like cosmic rotating and vibrating molecule. On the other hand, if this inequality is reversed, the equilibrium is unstable.

2.1 The Restricted Three-Body Problem.

We shall consider the restricted planar three-body system. Examples are

1. Sun-Jupiter-Asteroid/Space-probe
2. Sun-Earth-Spacecraft
3. Earth-Moon-Satellite.

Each system consists of two heavy masses M_1 and M_2, and a third body having such small mass, say m, that its gravitational influence on M_1 and M_2 is negligible.

2.1.1 The Starting Point: The Nature of Things in the Inertial Frame of the Fixed Stars

The mathematical formulation of these three-body problems rests on four interrelated properties:

1. The systems under consideration are those where the orbits of M_1 and M_2 are *circular* and the motion of m is *co-planar* with that of M_1 and M_2. This implies that the mass m is subjected to the gravitational force of two bodies whose separation vector

   $$\overrightarrow{M_1 M_2} \equiv \vec{a}(t) = a\,\vec{\alpha}(t) = a \begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} \tag{4}$$

   rotates, but does so rigidly with constant length

   $$|\vec{a}(t)| = \textit{constant} \equiv a.$$

2. By choosing to put the origin at the center of mass of M_1 and M_2, the location of the two masses is given by

   $$M_1: \quad \vec{R}_1(t) = -\frac{M_2}{M_1+M_2}\vec{a}(t) \equiv -\mu\, a\,\vec{\alpha}(t) \tag{5}$$

   $$M_2: \quad \vec{R}_2(t) = +\frac{M_1}{M_1+M_2}\vec{a}(t) \equiv (1-\mu)\, a\,\vec{\alpha}(t) \tag{6}$$

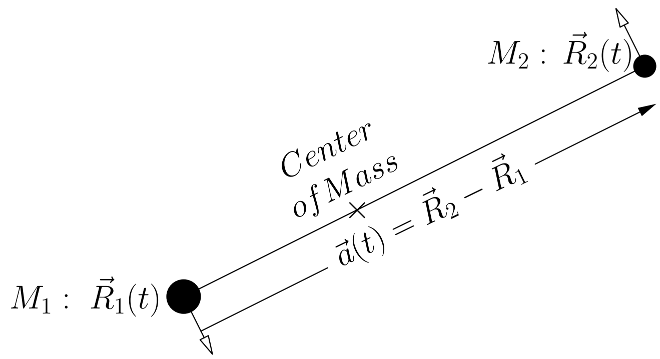


Figure 7: Two masses rotating around their center of mass

Here we have introduced the *fractional masses*

$$\mu = \frac{M_2}{M_1 + M_2} \qquad (\text{"}\mathit{Planet}\text{"}) \tag{7}$$

$$1 - \mu = \frac{M_1}{M_1 + M_2} \qquad (\text{"}\mathit{Sun}\text{"}) \tag{8}$$

of the two respective masses M_2 and M_1.

3. The angular velocity with which M_1 and M_2 orbit their center of mass is determined by applying mass$\times$(centripetal acceleration) = gravitational force to either mass. Newton's Third Law together with Figure 8 guarantee that the results will be the same. One obtains

$$M_1 \omega^2 \frac{M_2}{M_1 + M_2} a = \frac{G\, M_1 M_2}{a^2}.$$

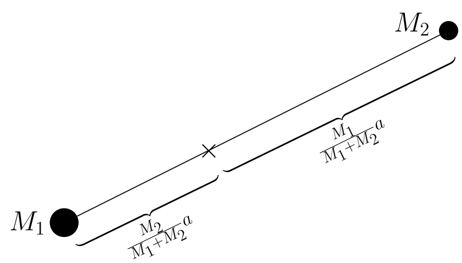


Figure 8: Orbital radii of M_1 and M_2

and hence

$$\boxed{G\,(M_1 + M_2) = \omega^2 a^3}. \tag{9}$$

This is the *1-2-3 law,* also known as Kepler's generalized Third Law. It says that once one has measured the period $\frac{2\pi}{\omega}$ and the size a of a binary system, its total mass $M_1 + M_2$ is known and determined.

4. The time-periodic potential energy of a mass m located at $\vec{r}(x, y, z)$ is

$$P.E.(\vec{r}) = -\frac{GM_2 m}{|\vec{R}_2(t) - \vec{r}|} - \frac{GM_1 m}{|\vec{R}_1(t) - \vec{r}|} \tag{10}$$

$$= -\frac{G\,(M_1 + M_2)m}{a}\left(\underbrace{\frac{\mu}{|(1-\mu)\vec{\alpha}(t) - \vec{r}/a|}}_{\equiv\rho_2} + \underbrace{\frac{1-\mu}{|-\mu\vec{\alpha}(t) - \vec{r}/a|}}_{\equiv\rho_1}\right) \tag{11}$$

$$= -\omega^2 a^2 m\left(\frac{\mu}{\rho_2} + \frac{1-\mu}{\rho_1}\right) \tag{12}$$

where we used Eqs.(5-6), (7-8), and (9) repectively.

2.1.2 Transformation into the Corotating Coordinate Frame

We now focus on the motion of the body m relative to the frame corotating with M_1 and M_2. A point $\vec{x}_{rot} = \begin{pmatrix} x_{rot} \\ y_{rot} \end{pmatrix}$ fixed in this frame will move relative to the fixed stars according to

$$\vec{r} \equiv \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\begin{pmatrix} x_{rot} \\ y_{rot} \end{pmatrix} \equiv T(t)\,\vec{x}_{rot}. \tag{13}$$

The task of applying this transformation to the to-be-minimized Lagrangian integral

$$\int (K.E. - P.E.)\,dt$$

consists of the five steps below. The final result is given by Eq.(18)

Step (i)
The trajectory of the body relative to the corotating frame, $\vec{x}_{rot}(t)$, is related to its trajectory $\vec{r}(t)$ relative to the fixed stars by

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = T(t)\,\vec{x}_{rot}(t). \tag{14}$$

Step (ii)
The separation vector $\overrightarrow{M_1 M_2}$, Eq.(4), has the same relation between its fixed-stars and its corotating frame representatives,

$$\vec{a}(t) = \begin{pmatrix} a\,\cos\omega t \\ a\,\sin\omega t \end{pmatrix}_{fixed} = T(t)\begin{pmatrix} a \\ 0 \end{pmatrix}_{rot}. \tag{15}$$

Thus, while in the fixed frame the separation vector $\overrightarrow{M_1M_2}$ rotates, in the rotating frame it remains fixed lying along the x_{rot}-axis. In other words, in the rotating coordinate frame the bodies M_1 and M_2 remain statically situated along the x_{rot}-axis.

The static nature of the bodies M_1 and M_2 in the rotating coordinate frame is depicted in Figure 9 . There, for subsequent mathematical efficiency, the coordinates have been scaled in terms of the constant M_1M_2-separation a.

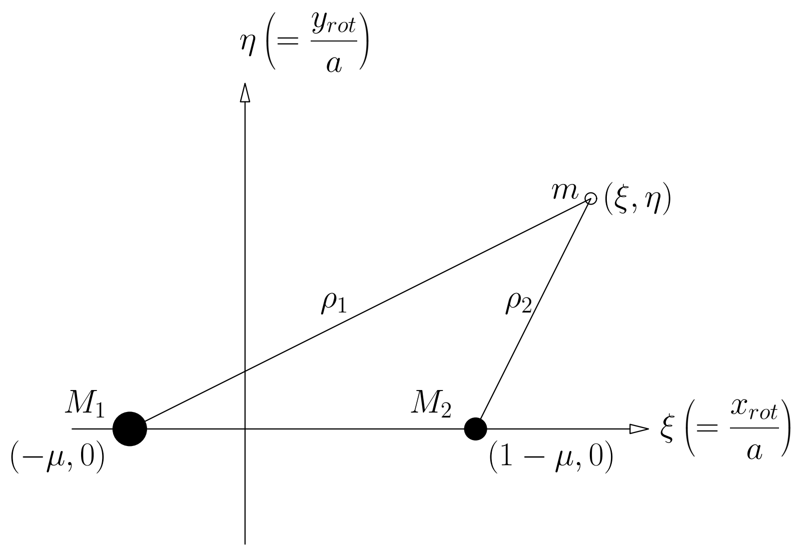


Figure 9: Rotating coordinate frame in which bodies M_1 and M_2 remain situated along its horizontal axis. In this frame they provide a static gravitational field for the dynamics of the body m.

Step (iii)
Whereas relative to the fixed-stars frame the potential energy function $P.E.$, Eq.(11), is a periodic function of time, in the rotating frame it is static. From a physics perspective this is obvious[2]. From a mathematical perspective this is a consequence of the orthogonality of the time-dependent point transformation T, Eqs.(13) and (15),

$$T(t):\left\{\begin{array}{rl}\vec{x}_{rot}\longrightarrow & T(t)\vec{x}_{rot}=\vec{r}\\ \vec{a}_0\longrightarrow & T(t)\vec{a}_0=\vec{a}(t)\end{array}\right. .$$

[2]In a merry-go-round corotating with two masses about their center of mass the laws of physics during one interval of time are the same as those during a later interval of time. For example, the spinning earth with mountain masses on opposite hemispheres make up such a merry-go-round.

Applying it to the arguments of the distance from m to M_1 and M_2, namely

$$\begin{aligned}\rho_1\left(\vec{a(t)},\vec{r}\right) &= \quad |-\mu\vec{a}(t)-\vec{r}\,| \\ \rho_2\left(\vec{a(t)},\vec{r}\right) &= |(1-\mu)\vec{a}(t)-\vec{r}|\end{aligned}$$

yields new functions[3]. Their domain is the rotating frame, and they are given by

$$\begin{aligned}\rho_1\circ T(t)\left(\vec{a}_0,\vec{x}_{rot}\right) &= |-\mu T(t)\vec{a}_0 - T(t)\vec{x}_{rot}| \\ &= \left|-\mu T(t)\begin{pmatrix} a \\ 0\end{pmatrix} - T(t)\begin{pmatrix} x_{rot} \\ y_{rot}\end{pmatrix}\right| \\ &= \left|-\mu\begin{pmatrix} a \\ 0\end{pmatrix} - \begin{pmatrix} x_{rot} \\ y_{rot}\end{pmatrix}\right| \\ &= a\sqrt{\left(\mu+\frac{x_{rot}}{a}\right)^2+\left(\frac{y_{rot}}{a}\right)^2} \\ &= a\sqrt{(\mu+\xi)^2+\eta^2}\end{aligned}$$

and

$$\begin{aligned}\rho_1\circ T(t)\left(\vec{a}_0,\vec{x}_{rot}\right) &= |(1-\mu)T(t)\vec{a}_0 - T(t)\vec{x}_{rot}| \\ &= a\sqrt{(\xi+1-\mu)^2+\eta^2}.\end{aligned}$$

These new functions depend only on the dimensionless rotating frame coordinates

$$\begin{aligned}\xi &= \frac{x_{rot}}{a} \\ \eta &= \frac{y_{rot}}{a}.\end{aligned}$$

Thus *relative to the rotating frame* the gravitational potential energy, Eq.(12), is

$$P.E._{grav.\,rot.\,frame} = -\omega^2 a^2 m\left(\frac{\mu}{\sqrt{(\xi+\mu)^2+\eta^2}}+\frac{1-\mu}{\sqrt{(\xi+1-\mu)^2+\eta^2}}\right). \tag{16}$$

It is independent of time and depends only on the position (ξ,η) of the body m.

Step (iv)

[3]the "pull backs" of ρ_i by T

Consider the kinetic energy of m in the fixed-stars (inertial) frame:

$$\begin{aligned} K.E. &= \frac{m}{2}\dot{\vec{r}}\cdot\dot{\vec{r}} \\ &= \frac{m}{2}\left(T\vec{x}_{rot}\right)^{\cdot}\cdot\left(T\vec{x}_{rot}\right)^{\cdot} \\ &= \frac{m}{2}\left(\dot{\vec{x}}^{t}_{rot}T^{t}+\vec{x}^{t}_{rot}\dot{T}^{t}\right)\left(T\dot{\vec{x}}_{rot}+\dot{T}\vec{x}_{rot}\right) \end{aligned}$$

Here the superscript "t" indicates transpose. For the rotation matrix T, Eq.(13), one has

$$\begin{aligned} T^{t}T &= I \\ \dot{T}^{t}\dot{T} &= \omega^{2}I \\ \dot{\vec{x}}^{t}_{rot}T^{t}\dot{T}\vec{x}_{rot}+\vec{x}^{t}_{rot}\dot{T}^{t}T\dot{\vec{x}}_{rot} &= 2\vec{x}^{t}_{rot}\dot{T}^{t}T\dot{\vec{x}}_{rot} \\ &= 2\omega\vec{x}^{t}_{rot}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\dot{\vec{x}}_{rot} \\ &= 2\omega\, a^{2}\left(\xi\dot{\eta}-\eta\dot{\xi}\right) \end{aligned}$$

Consequently, relative to the rotating frame, the kinetic energy decomposes into three parts:

$$K.E. = \underbrace{\frac{1}{2}ma^{2}(\dot{\xi}^{2}+\dot{\eta}^{2})}_{rot.K.E.}+\underbrace{m\omega\, a^{2}(\xi\dot{\eta}-\eta\dot{\xi})}_{C.E.}+\underbrace{\frac{1}{2}m\omega^{2}a^{2}(\xi^{2}+\eta^{2})}_{W_{rot}}. \qquad (17)$$

From the perspective of physics it is worthwhile to identify them individually:

1. The rotational kinetic energy, $rot.K.E.$, relative to the rotating frame.
2. The *Coriolis energy*, $C.E.$, which is the amount of energy in the inertial frame necessary to speed up m's angular velocity[4]from zero to ω in the inertial frame.

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[4]Obtained by introducing polar coordinates $a\xi = r\cos\theta$, $a\eta = r\sin\theta$. For pure θ-motion consider the rotational version of Newton's Second Law,

$$torque = mr^{2}\ddot{\theta}.$$

The work performed my this torque as it acts over an angular displacement $(\omega+\dot{\theta})\,dt$ increases the energy of an orbiting mass m by an amount

$$d(energy) = mr^{2}\ddot{\theta}[\omega+\dot{\theta}]\,\dot{d}t.$$

Consequently, the total amount of inertial energy imparted to m is $\int(torque)\,(angle\ swept\ out\ by\ m\ in\ the\ inertial\ frame\ during\ time\ dt) = \int(mr^{2}\ddot{\theta})([\omega+\dot{\theta}]dt)$ $= mr^{2}\dot{\theta}\omega + mr^{2}\dot{\theta}^{2}/2$. This is the sum of two partial energies, namely, (i) the *Coriolis Energy* $C.E. = mr^{2}\dot{\theta}\,\omega = ma^{2}(\xi\dot{\eta}-\eta\dot{\xi})$, which is the middle term of Eq.(17), and (ii) $rot.\ K.E. = mr^{2}\dot{\theta}^{2}/2$, which is the additional inertial work necessay to give m non-zero angular velocity $\dot{\theta}$ in the rotating frame.

3. The *kinetic work function*, W_{rot}, which is the amount of inertial kinetic energy necessay to move m from the origin to the location(ξ, η) in the rotating frame.

The Lagrangian integral to be minimized is therefore

$$\int (K.E. - P.E.)\, dt = \int (rot.K.E. + C.E. - rot.P.E.)\, dt \tag{18}$$

where

$$rot.P.E. = ma^2\omega^2\Phi(\xi, \eta)$$

and

$$\Phi(\xi, \eta) = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{\sqrt{(\xi+\mu)^2 + \eta^2}} + \frac{\mu}{\sqrt{(\xi+\mu+1)^2 + \eta^2}} \tag{19}$$

is the dimensionless scalar function in the rotating frame. It includes also the kinetic work function as an additive contribution to the (negative of the) two gravitational potentials.

2.1.3 The Equations of Motion

A necessary condition for the Lagrange integral to be minimized by $(\xi(t), \eta(t))$ is that the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} \qquad i = \xi, \eta$$

be satisfied. They are with the help of Eq.(19)

$$\begin{aligned} i = \xi : \quad \ddot{\xi} &= +2\omega\dot{\eta} + \omega^2 \partial_\xi \Phi \\ &= +2\omega\dot{\eta} - \omega^2 \left[\xi \underbrace{\left(-1 + \frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right)}_{f} + \mu(1-\mu)\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) \right] \end{aligned} \tag{20}$$

and

$$\begin{aligned} i = \eta : \quad \ddot{\eta} &= -2\omega\dot{\xi} + \omega^2 \partial_\eta \Phi \\ &= -2\omega\dot{\xi} - \omega^2 \left[\eta \underbrace{\left(-1 + \frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right)}_{f} \right] \end{aligned} \tag{21}$$

Here, for the sake of notational economy, we write

$$\rho_1 = \sqrt{(\xi+\mu)^2+\eta^2}$$
$$\rho_2 = \sqrt{(\xi+\mu-1)^2+\eta^2}$$

as the rescaled distance of m from M_1 and M_2in the rotating frame.

This pair of coupled differential equations is mathematically equivalent to the equations of motion of a negatively charged particle with charge-to-mass ratio

$$\frac{q}{m} = -\omega^2$$

moving in a planar electric field

$$\vec{E} = -\left(\vec{i}\cdot\partial_\xi\Phi + \vec{j}\cdot\partial_\eta\Phi\right) \equiv -\vec{\nabla}\Phi$$

combined with a constant magnetic field

$$\vec{B} = \vec{i}\cdot 0 + \vec{j}\cdot 0 + \vec{k}\cdot\frac{-2}{\omega}$$

which is perpendicular to the electric field and to the orbital plane. Identifying the particle trajectory as the moving vector

$$\vec{x}(t) = \vec{i}\cdot\xi(t) + \vec{l}\cdot\eta(t) + \vec{k}\cdot 0$$

one rewrites Eqs.(20)-(21) as

$$\begin{aligned}\ddot{\vec{x}} &= 2\omega\,\dot{\vec{x}}\times\vec{k} + \omega^2\vec{\nabla}\Phi \qquad (22)\\ &= \frac{q}{m}\,\dot{\vec{x}}\times\vec{B} + \frac{q}{m}\,\vec{E}.\end{aligned}$$

In physics these equations are recognized as the Lorentz equations of motion, Newton's equations in the context of electromagnetic forces. In engineering they are recognized as governing the operation of a *magnetron*, the heart of radar transmitters and microwave ovens.

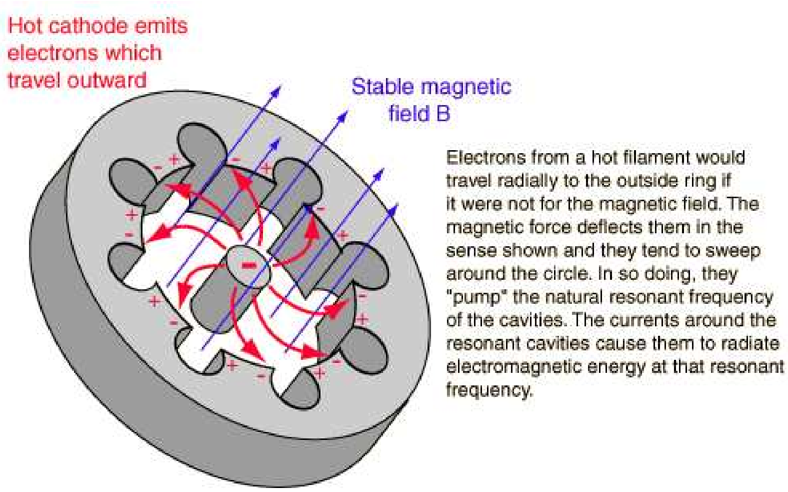
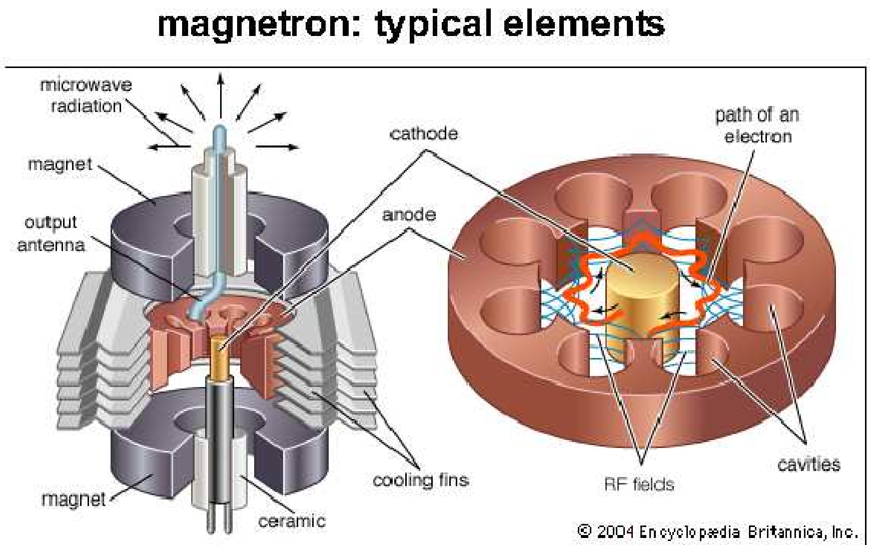


Figure 10: Charged particle trajectories in a magnetron.



The advantage of such a recognition is that it leads directly to an energy type integral. Indeed, multiply Eq.(22) by $\dot{\vec{x}}$ and obtain

$$\frac{d}{dt}\frac{1}{2}\left(\dot{\xi}^2+\dot{\eta}^2\right)=zero\;+\frac{d}{dt}\left(\omega^2\Phi\right)$$

It follows that

$$\frac{1}{2}\left(\dot{\xi}^2+\dot{\eta}^2\right)-\omega^2\Phi(\xi,\eta)\equiv H(\dot{\xi},\dot{\eta},\xi,\eta)\qquad \text{"Jacobi's integral"}$$

is an integral of motion, which was identified by Jacobi in 1836. The function H is a constant along any given trajectory that satisfies the equations of motion.

2.1.4 Energy Conservation in the Rotating Frame

The constancy of H along a trajectory is a statement of the conservation of energy of the third mass m in the rotating frame,

$$H(\dot{\xi},\dot{\eta},\xi,\eta)=(K.E.)_{rot}+(P.E.)_{rot}=const\equiv(T.E.)_{rot}\,. \tag{23}$$

Here

$$(K.E.)_{rot} = \frac{1}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right)$$

and[5]

$$\begin{aligned}(P.E.)_{rot} =& -\Phi(\xi,\eta)\\ =& -\frac{1-\mu}{\sqrt{(\xi+\mu)^2+\eta^2}} - \frac{\mu}{\sqrt{(\xi+\mu+1)^2+\eta^2}} - \frac{1}{2}(\xi^2+\eta^2).\end{aligned}$$

The roles of the energies which make up the energy law Eq.(23) are most efficiently captured by the simultaneous rendering of the two graphs, $(P.E.)_{rot}(\xi,\eta)$ and $(T.E.)_{rot}$ in Figure 11.

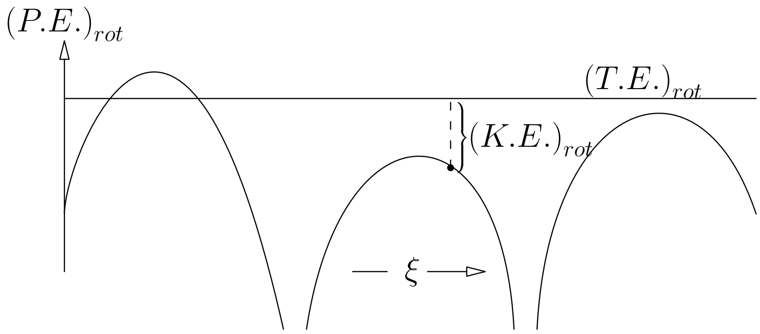


Figure 11: Graphs of $(P.E.)_{rot}(\xi,\eta)$ restricted to $\eta = 0$ and of $(T.E.)_{rot}$, which is constant. The classically allowed (forbidden) region is the one where the difference between $(T.E.)_{rot}$ and $(P.E.)_{rot}$, namely the kinetic energy $(K.E.)_{rot}$, is positive (negative). The allowed region is called the Hill region.

An unrestricted rendition of these graphs would be a 3-d graph, which, as in Figure 12, would extend over the whole (ξ,η)-plane. The motion of the particle, although in general quite irregular and even chaotic, would be deterministic and would be confined strictly to those regions for which $(K.E.)_{rot} \geq 0$. Their shape and topolgy depend on the value of $(T.E.)_{rot}$, and they are the shaded/green regions in Figure 13.

[5]The potential energy function $(P.E.)_{rot}(\xi,\eta)$ differs by a "mere" minus sign from the mathematical function $\Phi(\xi,\eta)$. However, in the hierarchy of concepts (i.e. from the perspective of epistemology, see e.g. "Introduction to Objectivist Epistemology" by Ayn Rand) $(P.E.)_{rot}$ – as introduced in physics with its minus sign – logically preceeds Φ: Before one can understand the meaning of Φ, one first has to understand the meaning of $(P.E.)_{rot}$. Physicists such as K.R. Symon (in his book "Mechanics") uses $(P.E.)_{rot}$, which he designates by $'V'(x,y)$. By contrast, mathematicians like J. Moser (in his "Lectures on Hamiltonian Dynamics") and V.I. Arnold et al (in their "Mathematical Aspects of Classical Mechanics") work with Φ, which they designate by $V(x,y)$ and $V(\xi,\eta)$, and which is easier to manipulate mathematically.

The latter is, of course, just a constant. The difference between the two is the kinetic energy $(K.E.)_{rot} = \frac{1}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right)$, which must never be negative. It follows that the motion of the particle m with a given total energy $(T.E.)_{rot}$ is restricted to only those (ξ, η)-regions which satisfy

$$(P.E.)_{rot}\,(\xi, \eta) \leq (T.E.)_{rot} \tag{24}$$

These regions are called *Hill* regions. For every $(T.E.)_{rot}$ there is one or more of these *allowed*

regions. Those (ξ, η)-regions which violate the energy condition, Eq.(24) are inaccessible. They are *classically forbidden*[6]. These regions are the unshaded ones in Figure 13.

2.1.5 The Jacobi Integral

The restricted planar three-body problem could be solved completely, if besides $H(\dot{\xi},\dot{\eta},\xi,\eta)$ one could identify another integral, say $F(\dot{\xi},\dot{\eta},\xi,\eta)$, which is functionally independent of H. If that were the case, then one could find the trajectories in a way that led to analytical solubility of the two-body problem.

However, even by itself, H does give very useful information. In particular, as we shall see, H identifies which regions are dynamically accessible and which are forbidden by classical mechanics.

The constancy of H, say

$$H(\dot{\xi},\dot{\eta},\xi,\eta) = constant \equiv h,$$

implies that the only accessible regions are those that satisfy

$$\begin{aligned} 0 \leqslant \frac{1}{2\omega^2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) &= h + \Phi(\xi, \eta) \\ &= h + \left[\frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right] \end{aligned} \tag{25}$$

These regions are known as Hill's regions. They are important for exploration by space probes which have limited amount of fuel.

Those regions where the inequality (25) is violated are *forbidden* by classical mechanics. Such regions are characterized by

$$h < -\left[\frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right].$$

[6]If the classical mechanics formulation of the particle motion is replaced by one in terms of wave (quantum) mechanics, then the allowed regions are those where the wave function of the particle oscillates. On the other hand, the classically forbidden regions are still accessible, but the wave function is decreasing exponentially so that the expectation value for of measuring the particle as present is also decreasing exponentially.

An unpropelled body may not penetrate such regions. The boundary(s) of these regions is the locus of points where the value h of the Jacobi integral is such that the velocity vanishes:

$$\dot{\xi}^2 + \dot{\eta}^2 = 0.$$

This condition is equivalent to a relation between h and the boundary points,

$$h = -\left[\frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right],$$

whose graph is exhibited in Figure 12.

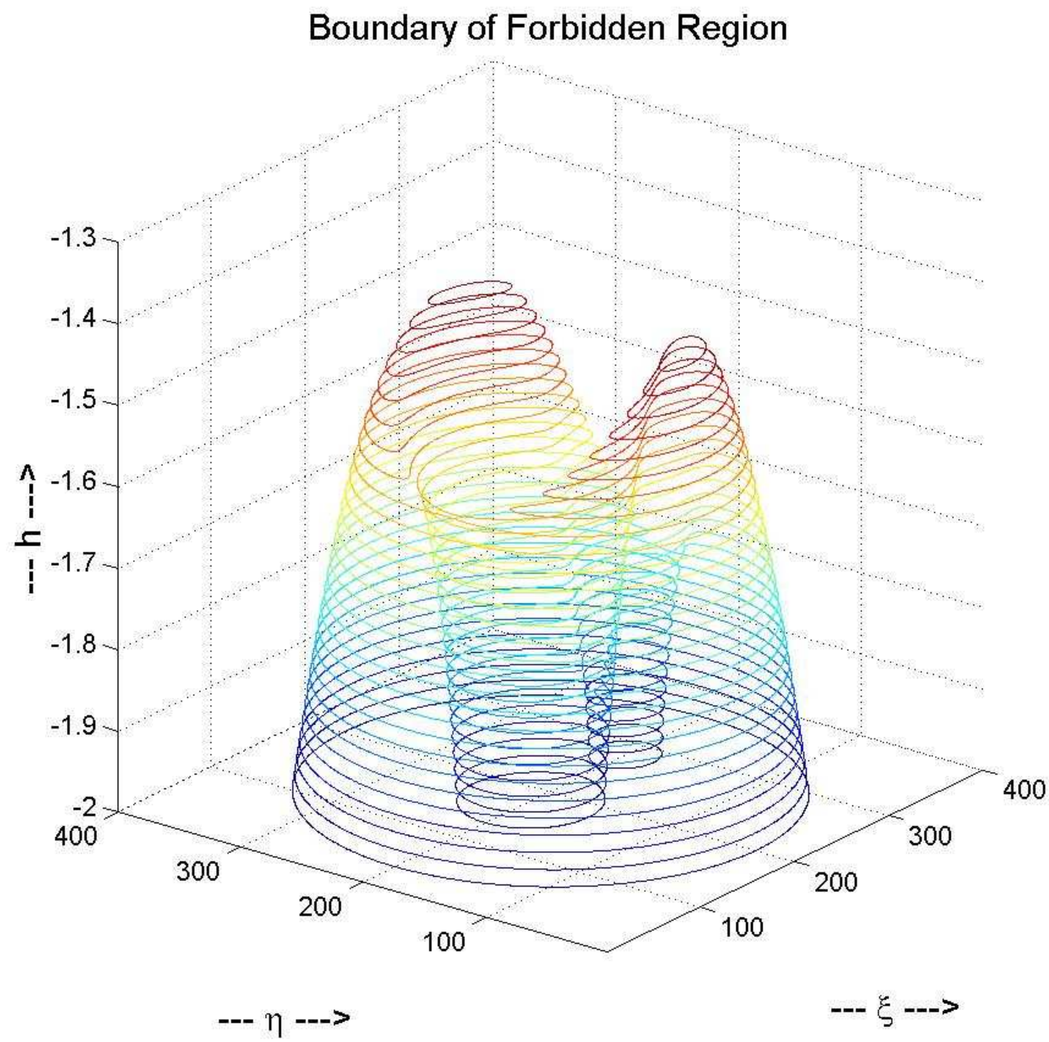


Figure 12: Boundary between forbidden and allowed regions parametrized by the value h of the Jacobi integral. At this boundary the velocity (in the rotating coordinate frame) of the body vanishes. The regions below the surface are forbidden, those above are allowed.

The intersection of this graph with each horizontal plane $h = constant$ consists of a boundary between the allowed and the forbidden region(s). In fact, these intersections form a parametrized family of boundaries. They are exhibited in Figure 13. Each of its panels is a horizontal slice through surface in Figure 12.

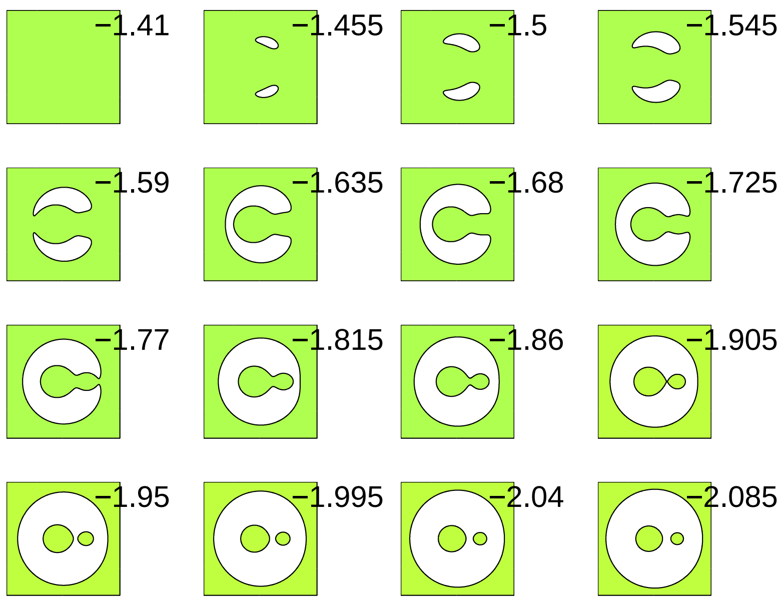


Figure 13: One parameter family of boundaries that separate the allowed (shaded/green) regions from those forbidden (unshaded/white) to an unpropelled body. The horizontal and vertical axes are those of ξ and η respectively. The numbers are the values of the Jacobi integral, which is the parameter.

The shape of the boundary(s) and the topology of the forbidden regions is determined by the most important attributes of the dynamical system, the location and the nature of its equilibrium trajectories. For these the body has zero velocity *and* zero acceleration,

$$\dot{\xi}^2 + \dot{\eta}^2 = 0 \quad and \quad \ddot{\xi} = \ddot{\eta} = 0.$$

This occurs where the effective potential $\Phi(\xi,\eta)$, Eq.(19), has its critical points,

$$\partial_\xi \Phi = \partial_\eta \Phi = 0.$$

In other words, because of Eqs.(20)-(21) one has the *equations for static equilibrium* in the rotating frame

$$\begin{aligned} 0 &= \partial_\xi \Phi \equiv \xi\, f + \mu(1-\mu)\left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3}\right) \\ 0 &= \partial_\eta \Phi \equiv \eta\, f, \end{aligned}$$

here

$$f = -1 + \frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3},$$

and again,

$$\begin{aligned} \rho_1 =& \sqrt{(\xi+\mu)^2+\eta^2} \\ \rho_2 =& \sqrt{(\xi+\mu-1)^2+\eta^2} \end{aligned}$$

2.1.6 Five Lagrange Points

The critical points of the effective potential fall into two classes, those for which $\eta = 0$ and those for which $\eta \neq 0$. Of the critical points with$\eta = 0$ there are three, L_1, L_2, and L_3. As one can see from Figure 14, they lie along the line connecting M_1 and M_2, and they are saddle points of $\Phi(\xi,\eta)$. Of the critical points with$\eta \neq 0$ there are two, L_4 and L_5. From the figures one sees that they form two equilateral triangles with M_1 and M_2, and they are maxima of $\Phi(\xi,\eta)$.

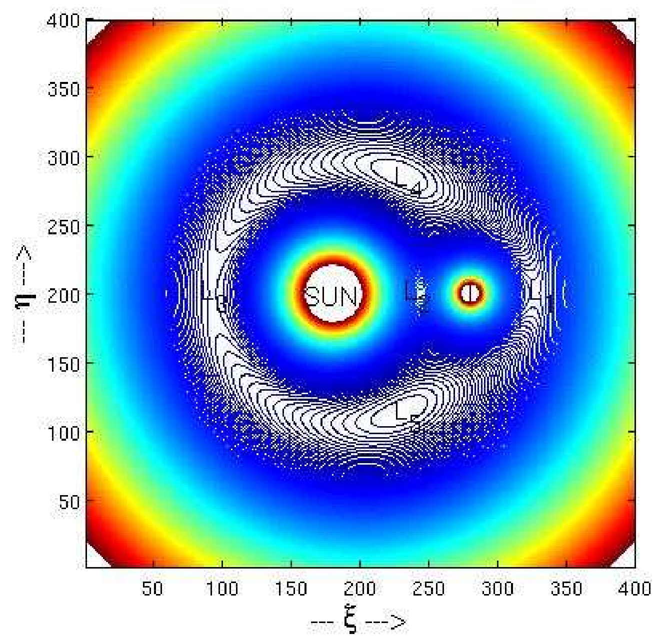


Figure 14: Isograms of equal h in the rotating coordinate frame spanned by ξ and η. L_1, L_2 and L_3 are the three unstable collinear Lagrange point. L_4 and L_5 are the two triangular point.

(i) Triangular Critical Points (L_4 and L_5).

For $\eta \neq 0$ the equations for static equilibrium are solved by

$$f = 0 \text{ and } \rho_1 = \rho_2 = 1,$$

and hence by

$$\xi = \frac{1}{2} - \mu, \quad \eta = \pm\frac{\sqrt{3}}{2}.$$

The location of the two critical points is therefore

$$L_4 = a\left(\frac{1}{2} - \mu, \, \frac{\sqrt{3}}{2}\right)$$
$$L_5 = a\left(\frac{1}{2} - \mu, \, \frac{-\sqrt{3}}{2}\right).$$

The masses M_1 and $M_2$are located at

$$M_1 : \quad (\xi, \eta) = (\,-a\mu, 0)$$
$$M_2 : \quad (\xi, \eta) = (a(1-\mu), 0)$$

It follows that $M_1M_2L_4$ and $M_1M_2L_5$ form two equilateral triangles whose sides equal a, the separation between M_1 and M_2.

The motion of particles (asteroids, spacecraft, etc.) in the neighborhood of L_4 and L_5 raises a number of key problems, namely the question of stability, of the existence of periodic solutions, and of integrals of motion. Answering these questions requires pushing back the frontier of mathematics. Juergen Moser has reported on this in his "Lectures on Hamiltonian Systems" (Memoirs of the American Mathematical Society, Number 81 (1968), P1-60, QA1 A527 No.80-81).

By specializing the linear stability criterion Eq.(3) on page 8 to the restricted ($M_3 \to 0$) three-body problem, one obtains

$$(1-\mu)\mu < \frac{1}{27}\,; \quad \mu = \frac{M_2}{M_1 + M_2}.$$

This inequality implies that one has linear stability whenever

$$0 < \mu < \mu_1 = .0385 \tag{26}$$

If the second mass, say Jupiter (in the the Sun-Jupiter-Achilles system) were to lie outside this interval, then the libration points L_4 and L_5 would be unstable. However, for the Sun_Jupiter system one has

$$\mu = \frac{M_2}{M_1 + M_2} = .000954.$$

Consequently, one concludes that the S-J-A is stable in the *linear* approximation. But linear stability ignores the nonlinear (quadratic, cubic, etc) contributions to the evolution of Achilles perturbed away from the critical point L_4 (or L_5) of $\Phi(\xi.\eta)$. If one takes all these contributions into account, one finds that the interval, Eq.(26), gets shrunk to

$$0 < \mu < \mu_{1c} = .0109.$$

In spite of this, Jupiter with its $\mu = .000954$ still satisfies the stronger inequality. Consequently, Achilles is not only linearly stable, but absolutely (non-linearly) stable as well.

(ii) Collinear Critical Points (L_1, L_2, L_3)

For $\eta = 0$ the only nontrivial equation for static equilibrium is the condition

$$\partial_\xi \Phi(\xi, \eta = 0) = 0.$$

This is a one-dimensional critical point problem for Φ evaluated along the ξ-axis. There one has

$$\Phi(\xi, \eta = 0) = -\frac{1}{2}\xi^2 - \frac{1-\mu}{\xi + \mu} - \frac{\mu}{\xi + \mu - 1}.$$

Finding the critical point of this function amounts to finding the roots of a quintic equation. That there are only three real roots (L_1, L_2, L_3) is evident from Figure 14. Alternatively, one arrives at the same conclusion by interpolating between the asymptotic behaviors of Φ as

$$\begin{aligned} \xi &\longrightarrow -\infty \\ \xi &\longrightarrow -\mu \\ \xi &\longrightarrow 1-\mu \\ \xi &\longrightarrow +\infty \end{aligned}$$

By inspection one sees that $\partial_\xi \partial_\xi \Phi < 0$ whenever Φ is defined. Consequently, Φ behaves as shown in Figure : there are only three critical point corresponding to L_1, L_2 and L_3.

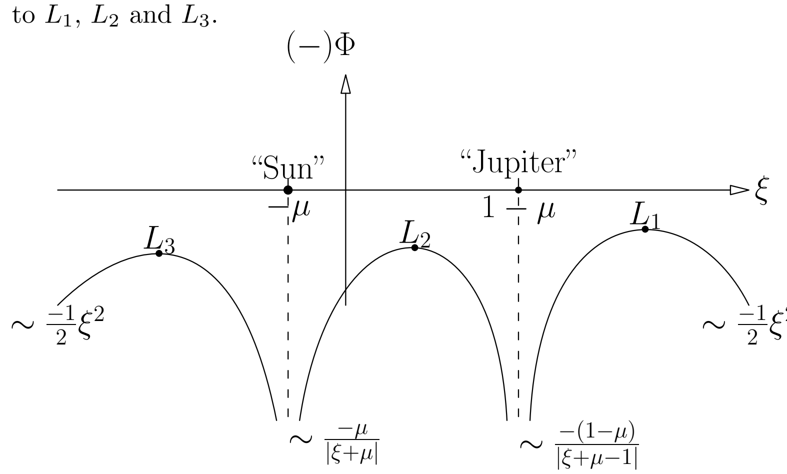


Figure 15: The three collinear unstable Lagrange libration points L_1, L_2 and L_3.They lie on the straight line passing through the two masses M_1 ("Sun") and M_2 ("Jupiter").