

Lecture 41-42.

The Linearized Einstein field equations:

1. Motivation
2. General formulation.
3. Gauge invariant geometrical formulation.

Appendix

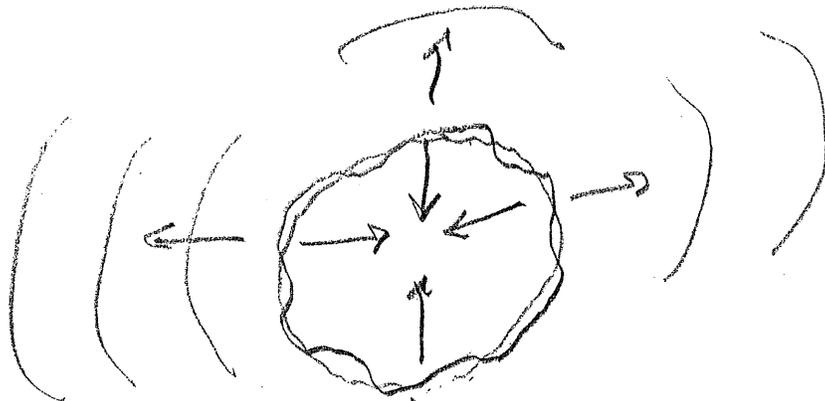
Elementary Perturbation Theory

on a Generic Spherically Symmetric

Space time

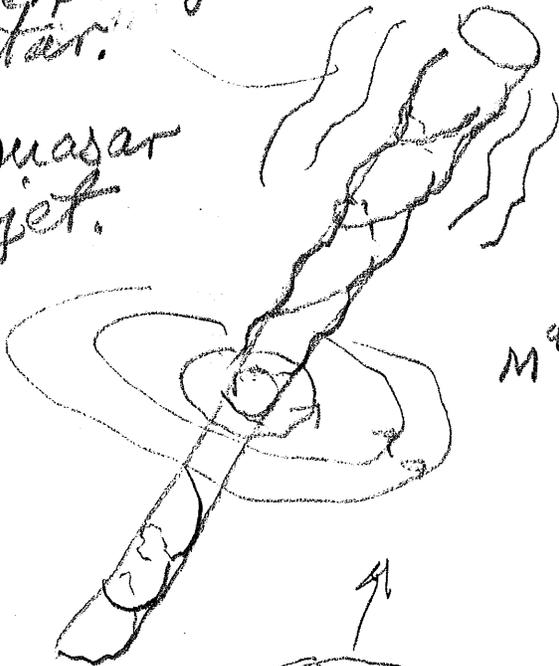
Dynamics of perturbed geometry and matter configuration ①

Evolution of perturbed geometry and matter configuration.

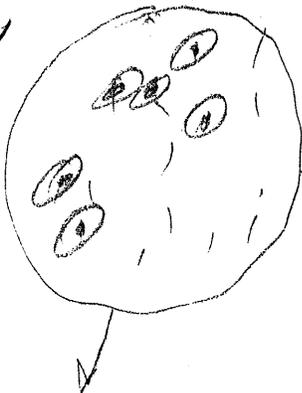


collapsing or exploding star.

quasar jet.



expanding/collapsing universe



Pert'n on a generic spherical spacetime

$$M^4: ds^2 = g_{AB} dx^A dx^B + \underbrace{r^2}_{M^2} \underbrace{(d\theta^2 + \sin^2\theta d\phi^2)}_{S^2}$$

$$M^2 = M^4 / S^2$$

perturbed spherical geometry

Pert'n on a generic cylindrical spacetime

perturbed cylindrical geometry

$$M^4: ds^2 = g_{AB} dx^A dx^B + \underbrace{r^2}_{M^2} \underbrace{(dz^2 + \psi d\phi^2 + \underbrace{r^2}_{M^2} dx^p dx^q)}_{S^1 \times S^2}$$

$$M^2 = M^4 / (R \times S^1)$$

perturbed homogeneous universe.

Pert'n on a generic homog. spacetime S^3

$$M^3 = M^4 / S^3$$

$$M^4: ds^2 = -a^2(\eta) d\eta^2 + a^2 \left[dx^2 + \underbrace{\sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)}_{S^3} \right]$$

$$M^3 = M^4 / S^3$$

SUMMARY

(2)

Unperturbed Einstein Field Equations

$$R_{\mu}^{\nu}[g_{\alpha\beta}] - \frac{1}{2} \delta_{\mu}^{\nu} R[g_{\alpha\beta}] = 8\pi T_{\mu}^{\nu}[p, \rho, u^{\alpha}, F_{\alpha\beta}, g_{\alpha\beta}]$$

Perturbed Einstein Field Eq's

$$R_{\mu}^{\nu}[g_{\alpha\beta} + \Delta g_{\alpha\beta}] - \frac{1}{2} \delta_{\mu}^{\nu} R[g_{\alpha\beta} + \Delta g_{\alpha\beta}] = 8\pi T_{\mu}^{\nu}[p + \Delta p, \rho + \Delta \rho, u^{\alpha} + \Delta u^{\alpha}, F_{\alpha\beta} + \Delta F_{\alpha\beta}, g_{\alpha\beta} + \Delta g_{\alpha\beta}]$$

Linearized Einstein Field Eq's:

With $\Delta R_{\mu}^{\nu} \equiv R_{\mu}^{\nu}[g_{\alpha\beta} + \Delta g_{\alpha\beta}] - R_{\mu}^{\nu}[g_{\alpha\beta}]$

$$\Delta R_{\mu}^{\nu} \equiv T_{\mu}^{\nu}[p + \Delta p, \dots, \text{etc} + \Delta \text{etc}] - T_{\mu}^{\nu}[p, \dots, \text{etc}]$$

one obtains

$$\Delta R_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} \Delta R = 8\pi \Delta T_{\mu}^{\nu}$$

Letting $\Delta g_{\alpha\beta} = h_{\alpha\beta}$

one finds

$$\left[-\frac{1}{2} h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - h^{\alpha}{}_{\mu;\nu;\alpha} - h^{\alpha}{}_{\nu;\mu;\alpha} + (h^{\alpha}{}_{\alpha})_{;\mu;\nu} + h_{\mu\nu} R + g_{\mu\nu} \{ h^{\sigma\rho}{}_{;\rho;\sigma} - h^{\alpha}{}_{\alpha;\beta}{}^{;\beta} - h^{\alpha\beta}{}_{R;\alpha\beta} \} \right] = 8\pi \Delta T_{\mu\nu}$$

The topics we aim to cover in these lectures are: ③ ④

1. Perturbed (i.e. linearized) Einstein field equation on an arbitrary space time.

a) Perturbation in metric

b) " " Christoffel symbols

c) " " Riemann-Christ. curvature

d) Infinitesimal coordinate (or "gauge") transformation, ^{tensor.}

e) Perturbations fictitious due to infinitesimal coord xformations.

skip & go to ⑤

2. The 2-dimensional reduced space-time manifold, M^2 , which is \perp to the concentric spheres.

4. Scalar, Vector, Tensor harmonics on S^2 .

3. Einstein and Maxwell eq'ns on a spherically symmetric space time.

5. Reduction of a perturbed tensor on 4-D space time to a set of geometrical ~~and~~ perturbation

objects on M^2 ,

(4)

6. Gauge invariant geometrical perturbation objects for

i, metric

ii, Maxwell

iii, stress-energy

iv, vector

7. Linearized Einstein & Maxwell Field eq'ns in terms of gauge invariant geometrical perturbation objects

a) odd parity

b) even parity

8. Formulation in terms of differential forms

9. Coupled modes and normal modes.

10. Outstanding mathematical and physics problems in relativistic perturbation theory

1. Linearized Einstein field equations

perturbations
in

metric
christoffel
Riemann
Ricci
curvature int.
Einstein
coordinate
system

(5)

a) Perturbation in Metric tensor.

i. Consider a space-time endowed with the metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Alternatively, consider the perturbed metric

$$\bar{ds}^2 = \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$= [g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x)] dx^\mu dx^\nu$$

on another copy of the same manifold.

ii. The symmetric tensor field

$$\Delta g_{\mu\nu}(x) dx^\mu dx^\nu \equiv h_{\mu\nu}(x) dx^\mu dx^\nu$$

is called the perturbation ~~and~~ away from the background metric $g_{\mu\nu} dx^\mu dx^\nu$.

More briefly we shall refer to the perturbation

$$h_{\mu\nu}(x)$$

iii. Properties:

$$h^{\mu}_{\nu} = h_{\sigma\nu} g^{\sigma\mu}$$

$$h^{\mu\nu} = h_{\sigma\tau} g^{\sigma\mu} g^{\tau\nu} = -\Delta g^{\mu\nu}$$

this last equality follows from

⑥

$$\begin{aligned}
 g_{\mu\sigma} g^{\sigma\nu} &= \delta_{\mu}^{\nu} \\
 (g_{\mu\sigma} + h_{\mu\sigma})(g^{\sigma\nu} + \Delta g^{\sigma\nu}) &= \delta_{\mu}^{\nu} \quad \left. \begin{array}{l} \text{ignoring 2nd order term} \\ \Rightarrow h_{\mu}^{\nu} + g_{\mu\sigma} \Delta g^{\sigma\nu} = 0 \end{array} \right\} \\
 &\text{or } \boxed{h^{\mu\nu} = -\Delta g^{\mu\nu}} \quad \text{Q.E.D.}
 \end{aligned}$$

b) Perturbed Christoffel symbols

Christoffel symbol for unperturbed space-time is

$$\Gamma_{\alpha\beta}^{\delta} = \frac{1}{2} g^{\delta\sigma} (g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})$$

Here comma denotes partial derivative with respect to the indexed coordinate

$$\begin{aligned}
 \text{Now: } \Delta(g_{\sigma\alpha,\beta}) &= \frac{\partial \bar{g}_{\sigma\alpha}(x)}{\partial x^{\beta}} - \frac{\partial g_{\sigma\alpha}(x)}{\partial x^{\beta}} \\
 &= \frac{\partial \Delta g_{\sigma\alpha}(x)}{\partial x^{\beta}} = h_{\sigma\alpha,\beta}
 \end{aligned}$$

$$\text{also } \Delta g^{\delta\sigma} = -h^{\delta\sigma}$$

$$\begin{aligned}
 \boxed{\Delta \Gamma_{\alpha\beta}^{\delta}} &= \bar{\Gamma}_{\alpha\beta}^{\delta} - \Gamma_{\alpha\beta}^{\delta} \\
 &= -\frac{1}{2} h^{\delta\sigma} (g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma}) \\
 &\quad + \frac{1}{2} g^{\delta\sigma} (h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma}) \\
 &= \frac{1}{2} (h^{\delta}_{\alpha;\beta} + h^{\delta}_{\beta;\alpha} - h_{\alpha\beta}{}^{\delta}{}_{;\delta})
 \end{aligned}$$

THUS, $\Gamma_{\alpha\beta}^{\delta}$, $\Delta \Gamma_{\alpha\beta}^{\delta}$ are the COMPONENTS OF A TENSOR!

here the semi colon denotes covariant derivative; some of whose relevant properties are;

$$h_{\alpha\beta;\sigma} = h_{\alpha\beta,\sigma} - \Gamma_{\alpha\sigma}^{\rho} h_{\rho\beta} - \Gamma_{\beta\sigma}^{\rho} h_{\alpha\rho}$$

$$g_{\alpha\beta;\sigma} = 0 = g^{\alpha\beta}{}_{;\sigma}$$

$$h^{\gamma}{}_{\alpha;\beta} = g^{\delta\sigma} (h_{\sigma\alpha;\beta}) = (g^{\delta\sigma} h_{\sigma\alpha})_{;\beta}$$

Exercise: show $\Delta \Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} (h^{\gamma}{}_{\alpha;\beta} + h^{\gamma}{}_{\beta;\alpha} - h_{\alpha\beta}{}^{;\gamma})$

c) Perturbation of components of Riemann.

The components of the ~~Riemann~~ Riemann-Christoffel curvature tensor are

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta} \Gamma^{\mu}{}_{\beta\gamma}$$

Its perturbations are

$$\begin{aligned} \Delta R^{\alpha}{}_{\beta\gamma\delta} &= \Delta \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Delta \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Delta \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\mu\gamma} \Delta \Gamma^{\mu}{}_{\beta\delta} \\ &\quad - \Delta \Gamma^{\alpha}{}_{\mu\delta} \Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\mu\delta} \Delta \Gamma^{\mu}{}_{\beta\gamma} \\ &= \underbrace{(\Delta \Gamma^{\alpha}{}_{\beta\delta;\gamma} - \Gamma^{\mu}{}_{\delta\gamma} \Delta \Gamma^{\alpha}{}_{\beta\mu})}_{\textcircled{1}} - \underbrace{(\Delta \Gamma^{\alpha}{}_{\beta\gamma;\delta} - \Gamma^{\mu}{}_{\gamma\delta} \Delta \Gamma^{\alpha}{}_{\beta\mu})}_{\textcircled{2}} \\ &= \Delta \Gamma^{\alpha}{}_{\beta\delta;\gamma} - \Delta \Gamma^{\alpha}{}_{\beta\gamma;\delta} \end{aligned}$$

after cancelling because $\Gamma^{\mu}{}_{\gamma\delta} = \Gamma^{\mu}{}_{\delta\gamma}$

(P)

Perturbed Riemann tensor components (cont'd)
 after using the formula $\Delta \Gamma_{\beta\delta}^{\alpha} = \frac{1}{2} (h^{\alpha}_{\beta;\delta} + h^{\alpha}_{\delta;\beta} - h_{\beta\delta}{}^{;\alpha})$
 are

$$\Delta R^{\alpha}_{\beta\delta\gamma} = \frac{1}{2} (h^{\alpha}_{\beta;\delta;\gamma} + h^{\alpha}_{\delta;\beta;\gamma} - h_{\beta\delta}{}^{;\alpha}{}_{;\gamma} - h^{\alpha}_{\beta;\gamma;\delta} - h^{\alpha}_{\gamma;\beta;\delta} + h_{\beta\gamma}{}^{;\alpha}{}_{;\delta})$$

d) Perturbation of components of Ricci tensor

The components of the Ricci-tensor are

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Its perturbations are

$$\Delta R_{\mu\nu} = \frac{1}{2} (h^{\alpha}_{\mu;\nu;\alpha} + h^{\alpha}_{\nu;\mu;\alpha} - h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \cancel{h^{\alpha}_{\mu;\alpha;\nu}} - \cancel{h^{\alpha}_{\alpha;\mu;\nu}} + \cancel{h_{\mu\alpha}{}^{;\alpha}{}_{;\nu}})$$

↖ cancel ↗

$$\Delta R_{\mu\nu} = -\frac{1}{2} h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \frac{1}{2} h^{\alpha}_{\mu;\nu;\alpha} + \frac{1}{2} h^{\alpha}_{\nu;\mu;\alpha}$$

Note: $\Delta R_{\mu\nu} = \Delta R_{\nu\mu}$ i.e.

the perturbation tensor is also symmetric

e) Perturbation of ~~g~~ curvature Invariant.

The curvature invariant is

$$R = g^{\mu\nu} R_{\mu\nu} \equiv R^{\mu}_{\mu}$$

Its perturbation is

$$\begin{aligned} \Delta R &= \Delta(g^{\mu\nu} R_{\mu\nu}) = \Delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \Delta R_{\mu\nu} \\ &= -h^{\mu\nu} R_{\mu\nu} + \frac{1}{2} h^{\mu}_{\mu}{}^{;\alpha}{}_{;\alpha} + \frac{2}{2} h^{\alpha\mu}{}_{;\mu}{}_{;\alpha} \end{aligned}$$

$$\Delta R = -h^{\alpha}_{\alpha}{}^{;\beta}{}_{;\beta} + h^{\sigma\rho}{}_{;\rho}{}_{;\sigma} - h^{\mu\nu} R_{\mu\nu} - \frac{1}{2} h^{\alpha}{}_{\alpha}{}^{;\mu}{}_{;\mu}$$

f) Perturbation of Einstein field eq/ns

The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

where $T_{\mu\nu}$ is the stress energy tensor

G " " gravitational constant

($\frac{1}{15\,000\,000}$ in c.g.s units)

c " " speed of light ($3 \times 10^{10} \frac{\text{cm}}{\text{sec}}$)

Let $T^*_{\mu\nu} = \frac{G}{c^4} T_{\mu\nu}$ be the stress energy tensor expressed in geometrical units $[\frac{1}{(\text{length})^2} = \frac{G}{c^4} \frac{\text{energy}}{(\text{length})^3}]$, Then drop

the eq.
Thus in geometrical units Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Perturbed (Eior "linearized") Einstein field equations are:

$$\Delta R_{\mu\nu} - \frac{1}{2} \Delta (g_{\mu\nu} R) = 8\pi \Delta T_{\mu\nu}$$

Use results from e) & d) and obtain

$$\begin{aligned}
& -\frac{1}{2} [h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - h^{\alpha}{}_{\mu;\nu;\alpha} - h^{\alpha}{}_{\nu;\mu;\alpha} + h^{\alpha}{}_{\alpha;\mu;\nu} \\
& + h_{\mu\nu} R + g_{\mu\nu} \{h^{\sigma\rho}{}_{;\rho;\sigma} - h^{\alpha}{}_{\alpha;\beta}{}^{;\beta} - h^{\mu\nu} R_{\mu\nu}\}] = 8\pi \Delta T_{\mu\nu}
\end{aligned}$$