

MATH 5757: Appendix to Lecture 42

ELEMENTARY PERTURBATION THEORY
ON GENERIC
SPHERICALLY SYMMETRIC SPACETIMES

10 LECTURES

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References:

1. "Gauge Invariant Perturbations on Most General Spherically Symmetric Space-Times", Physical Review D 19, 2268-2272 (1979)
2. "Junction Conditions for Odd Parity Perturbations on Generic Spherically Symmetric Space-Times", Physical Review D 20, 3009-3014 (1979)
3. "Even Parity Junction Conditions for Perturbations on Most General Spherically Symmetric Space-Times", Journal of Mathematical Physics 20, 2540-2546 (1979)
4. "Invariant Coupled Gravitational, Acoustical, and Electromagnetic Modes on Most General Spherical Space-Times", Physical Review D 22, 1300-1312 (1980)
5. "Homogeneous Collapsing Star: Tensor and Vector Harmonics for Matter and Field Asymmetries", Physical Review D 18, 1773-1784 (1978)

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(ODD PARITY)

Perturbation Theory of Spherically Symmetric ^① Space Times.

Spherical symmetry plays a central role in the description of many phenomena of nature. This importance derives not so much from the fact that many phenomena are perfectly spherically symmetric but rather from the fact that many more are only spherically symmetric.

For certain well-defined reasons slight asymmetries are of fundamental interest in physics. This is particularly true in astrophysics where many fascinating phenomena are described by the slight asymmetries themselves.

However, the basic applicable equations are often too difficult to solve exactly.

Consequently, one resorts to linearized versions (2)
of these equations and thereby describes these
asymmetric phenomena in the limit of them
being negligibly small. Nevertheless, even
within this simplifying assumption workers have
found that obstacles can be time consuming
and can slow down ^{considerably} progress towards finding
interesting results. Indeed one sometimes
even hears, and for good reasons, that calculations
in perturbation theory are typically long,
tedious and filled with long mathematical expressions
and at times are down right messy. One of
~~the purp~~. Indeed the problem has at times become
so severe that perturbational computations
(the task of doing) has been assigned to
electronic computers who then proceed to

③

manipulate the algebraic expressions in order to bring them into a form where they can be analyzed numerically, again using a computer.

It is a purpose of these lectures to show that contrary to the above assessment, ~~perturbation theory~~ ~~pertaining~~ the theory of perturbations away from spherical symmetry is actually highly economical, versatile, and downright beautiful. The criterion of beauty here is that of Elie Cartan who considered indices in differential geometry a "debauchery," ~~to~~ to say nothing about coordinate dependent expressions.

The topics we aim to cover in these lectures are: ④

1. Perturbed (i.e. linearized) Einstein field equation on an arbitrary space time.
 - a) Perturbation in metric
 - b) " " Christoffel symbols
 - c) " " Riemann-Christ. curvature tensor.
 - d) Infinitesimal coordinate (or "gauge") transformation.
 - e) Perturbations [fictitious] due to infinitesimal coord xformations.
2. The 2-dimensional reduced space-time manifold M^2 , which is \perp to the concentric spheres.
4. Scalar, Vector, Tensor harmonics on S^2 .
3. Einstein and Maxwell eq'ns on a spherically symmetric space time.
5. Reduction of a perturbed tensor on 4-D space time to a set of geometrical ~~and~~ perturbation

objects on M^2 ,

(4)

6. Gauge invariant geometrical perturbation objects for

i, metric

ii, Maxwell

iii, stress-energy

iv, vector

7. Linearized Einstein & Maxwell Field eq'ns in terms of gauge invariant geometrical perturbation objects

a) odd parity

b) even parity

8. Formulation in terms of differential forms

9. Coupled modes and normal modes.

10. Outstanding mathematical and physics problems in relativistic perturbation theory

1. Linearized Einstein field equations

(5)

a) Perturbation in Metric tensor.

i. Consider a space-time endowed with the metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Alternatively, consider the perturbed metric

$$\bar{ds}^2 = \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$= [g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)] dx^\mu dx^\nu$$

on another copy of the same manifold.

ii. The symmetric tensor field

$$\delta g_{\mu\nu}(x) dx^\mu dx^\nu \equiv h_{\mu\nu}(x) dx^\mu dx^\nu$$

is called the perturbation ~~and~~ away from the background metric $g_{\mu\nu} dx^\mu dx^\nu$.

More briefly we shall refer to the perturbation

$$h_{\mu\nu}(x).$$

iii. Properties:

$$h^{\mu}_{\nu} = h_{\sigma\nu} g^{\sigma\mu}$$

$$h^{\mu\nu} = h_{\sigma\tau} g^{\sigma\mu} g^{\tau\nu} = -\Delta g^{\mu\nu}$$

this last equality follows from

⑥

$$\left. \begin{aligned} g_{\mu\sigma} g^{\sigma\nu} &= \delta_{\mu}^{\nu} \\ (g_{\mu\sigma} + h_{\mu\sigma})(g^{\sigma\nu} + \Delta g^{\sigma\nu}) &= \delta_{\mu}^{\nu} \end{aligned} \right\} \text{ignoring 2nd order term.}$$

$$\Rightarrow h_{\mu}^{\nu} + g_{\mu\sigma} \Delta g^{\sigma\nu} = 0$$

$$\text{or } \boxed{h^{\mu\nu} = -\Delta g^{\mu\nu}} \text{ Q.E.D.}$$

b) Perturbed Christoffel symbols

Christoffel symbol for unperturbed space-time is

$$\Gamma_{\alpha\beta}^{\delta} = \frac{1}{2} g^{\delta\sigma} (g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma})$$

Here comma denotes partial derivative with respect to the indexed coordinate

$$\begin{aligned} \text{Now: } \Delta(g_{\sigma\alpha,\beta}) &= \frac{\partial \bar{g}_{\sigma\alpha}(x)}{\partial x^{\beta}} - \frac{\partial g_{\sigma\alpha}(x)}{\partial x^{\beta}} \\ &= \frac{\partial \Delta g_{\sigma\alpha}(x)}{\partial x^{\beta}} = h_{\sigma\alpha,\beta}
 \end{aligned}$$

$$\text{also } \Delta g^{\delta\sigma} = -h^{\delta\sigma}$$

$$\begin{aligned} \Delta \Gamma_{\alpha\beta}^{\delta} &= \bar{\Gamma}_{\alpha\beta}^{\delta} - \Gamma_{\alpha\beta}^{\delta} \\ &= -\frac{1}{2} h^{\delta\sigma} (g_{\sigma\alpha,\beta} + g_{\sigma\beta,\alpha} - g_{\alpha\beta,\sigma}) \\ &\quad + \frac{1}{2} g^{\delta\sigma} (h_{\sigma\alpha,\beta} + h_{\sigma\beta,\alpha} - h_{\alpha\beta,\sigma}) \\ &= \frac{1}{2} (h^{\delta}_{\alpha;\beta} + h^{\delta}_{\beta;\alpha} - h_{\alpha\beta}{}^{\delta}{}_{;\sigma})
 \end{aligned}$$

THUS, UNLIKE $\Gamma_{\alpha\beta}^{\delta}$, $\Delta \Gamma_{\alpha\beta}^{\delta}$ are the COMPONENTS OF A TENSOR!

here the semi colon denotes covariant derivative
 some of whose relevant properties are:

$$h_{\alpha\beta;\sigma} = h_{\alpha\beta,\sigma} - \Gamma_{\alpha\sigma}^{\rho} h_{\rho\beta} - \Gamma_{\beta\sigma}^{\rho} h_{\alpha\rho}$$

$$g_{\alpha\beta;\sigma} = 0 = g^{\alpha\beta}{}_{;\sigma}$$

$$h^{\gamma}{}_{\alpha;\beta} = g^{\delta\sigma} (h_{\sigma\alpha;\beta}) = (g^{\delta\sigma} h_{\sigma\alpha})_{;\beta}$$

Exercise: show $\Delta \Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} (h^{\gamma}{}_{\alpha;\beta} + h^{\gamma}{}_{\beta;\alpha} - h_{\alpha\beta}{}^{;\delta})$

c) Perturbations of components of Riemann.

The components of the ~~Riemann~~ Riemann-Christoffel curvature tensor are

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta} \Gamma^{\mu}{}_{\beta\gamma}$$

Its perturbations are

$$\begin{aligned} \Delta R^{\alpha}{}_{\beta\gamma\delta} &= \Delta \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Delta \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Delta \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\beta\delta} + \Gamma^{\alpha}{}_{\mu\gamma} \Delta \Gamma^{\mu}{}_{\beta\delta} \\ &\quad - \Delta \Gamma^{\alpha}{}_{\mu\delta} \Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\mu\delta} \Delta \Gamma^{\mu}{}_{\beta\gamma} \\ &= \underbrace{(\Delta \Gamma^{\alpha}{}_{\beta\delta;\gamma} - \Gamma^{\mu}{}_{\delta\gamma} \Delta \Gamma^{\alpha}{}_{\beta\mu})}_{\textcircled{1}} - \underbrace{(\Delta \Gamma^{\alpha}{}_{\beta\gamma;\delta} - \Gamma^{\mu}{}_{\gamma\delta} \Delta \Gamma^{\alpha}{}_{\beta\mu})}_{\textcircled{2}} \\ &= \Delta \Gamma^{\alpha}{}_{\beta\delta;\gamma} - \Delta \Gamma^{\alpha}{}_{\beta\gamma;\delta} \end{aligned}$$

after cancelling
 because $\Gamma^{\mu}{}_{\gamma\delta} = \Gamma^{\mu}{}_{\delta\gamma}$

Perturbed Riemann tensor components (cont'd)
 after using the formula $\Delta \Gamma_{\beta\delta}^{\alpha} = \frac{1}{2}(h^{\alpha}_{\beta;\delta} + h^{\alpha}_{\delta;\beta} - h_{\beta\delta}{}^{;\alpha})$
 are

$$\Delta R^{\alpha}_{\beta\delta\gamma} = \frac{1}{2} (h^{\alpha}_{\beta;\delta;\gamma} + h^{\alpha}_{\delta;\beta;\gamma} - h_{\beta\delta}{}^{;\alpha}{}_{;\gamma} - h^{\alpha}_{\beta;\gamma;\delta} - h^{\alpha}_{\gamma;\beta;\delta} + h_{\beta\gamma}{}^{;\alpha}{}_{;\delta})$$

d) Perturbation of components of Ricci tensor

The components of the Ricci-tensor are

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

Its perturbations are

$$\Delta R_{\mu\nu} = \frac{1}{2} (h^{\alpha}_{\mu;\nu;\alpha} + h^{\alpha}_{\nu;\mu;\alpha} - h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \cancel{h^{\alpha}_{\mu;\alpha;\nu}} - \cancel{h^{\alpha}_{\alpha;\mu;\nu}} + \cancel{h_{\mu\alpha}{}^{;\alpha}{}_{;\nu}})$$

↖ cancel ↗

$$\Delta R_{\mu\nu} = -\frac{1}{2} h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \frac{1}{2} h^{\alpha}_{\mu;\nu;\alpha} + \frac{1}{2} h^{\alpha}_{\nu;\mu;\alpha}$$

Note: $\Delta R_{\mu\nu} = \Delta R_{\nu\mu}$ i.e

$$-\frac{1}{2} h^{\alpha}_{\alpha;\mu;\nu}$$

the perturbation tensor is also symmetric

e) Perturbation of g curvature Invariant.

The curvature invariant is

$$R = g^{\mu\nu} R_{\mu\nu} \equiv R^{\mu}_{\mu}$$

Its perturbation is

$$\begin{aligned} \Delta R &= \Delta(g^{\mu\nu} R_{\mu\nu}) = \Delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \Delta R_{\mu\nu} \\ &= -h^{\mu\nu} R_{\mu\nu} + \frac{1}{2} h^{\mu}{}_{\mu;\alpha}{}^{\alpha} + \frac{2}{2} h^{\alpha\mu}{}_{;\mu;\alpha} \end{aligned}$$

$$\Delta R = -h^{\alpha}{}_{\alpha;\beta}{}^{\beta} + h^{\sigma\sigma}{}_{;\beta;\sigma} - h^{\mu\nu} R_{\mu\nu} - \frac{1}{2} h^{\alpha}{}_{\alpha;\mu}{}^{\mu}$$

f) Perturbation of Einstein field eq'ns
The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \frac{G}{c^4} T_{\mu\nu}$$

where $T_{\mu\nu}$ is the stress energy tensor

G " " gravitational constant

($\frac{1}{15\,000\,000}$ in c.g.s units)

c " " speed of light ($3 \times 10^{10} \frac{\text{cm}}{\text{sec}}$)

Let $T_{\mu\nu}^* = \frac{G}{c^4} T_{\mu\nu}$ be the stress energy tensor expressed in geometrical units $[\frac{1}{(\text{length})^2} = \frac{G}{c^4} \frac{\text{energy}}{(\text{length})^3}]$, Then drop

the bar.

Thus in geometrical units Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

Perturbed (Eior "linearized") Einstein field equations are:

$$\Delta R_{\mu\nu} - \frac{1}{2} \Delta (g_{\mu\nu} R) = 8\pi \Delta T_{\mu\nu}$$

Use results from e) & d) and obtain

$$\begin{aligned}
& -\frac{1}{2} [h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - h^{\alpha}{}_{\mu;\nu;\alpha} - h^{\alpha}{}_{\nu;\mu;\alpha} + h^{\alpha}{}_{\alpha;\mu;\nu} \\
& + h_{\mu\nu} R + g_{\mu\nu} \{h^{\sigma\rho}{}_{;\rho;\sigma} - h^{\alpha}{}_{\alpha;\beta}{}^{;\beta} - h^{\mu\nu} R_{\mu\nu}\}] = 8\pi \Delta T_{\mu\nu}
\end{aligned}$$

Alternative Way of writing the Perturbed Einstein Field equations (106)

Recall that the Riemann tensor is defined by

$$(\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma)W = R(W, e_\gamma, e_\delta) = e_\alpha R^\alpha_{\beta\gamma\delta} w^\beta$$

Using semicolon notation on the vector $\xi = \xi^\alpha e_\alpha$

$$\xi_{\sigma;\gamma;\delta} - \xi_{\sigma;\delta;\gamma} = \xi_\beta R^\beta_{\sigma\gamma\delta}$$

This generalizes to a tensor $h = h^{\mu\nu} e_\mu \otimes e_\nu$

$$h_{\nu;\alpha;\beta} - h_{\mu\nu;\beta;\alpha} = h_{\mu\sigma} R^\sigma_{\nu\alpha\beta} + h_{\sigma\nu} R^\sigma_{\mu\alpha\beta}$$

Thus one has, for example,

$$\begin{aligned} h^\alpha_{\mu\nu;\alpha} &= h^\alpha_{\mu;\alpha;\nu} + h^\alpha_{\sigma} R^\sigma_{\mu\nu\alpha} + h_{\sigma\mu} R^{\sigma\alpha}_{\nu\alpha} \\ &= h^\alpha_{\mu;\alpha;\nu} + h^{\alpha\sigma} R_{\sigma\mu\nu\alpha} + h^\sigma_{\mu} R_{\sigma\nu} \end{aligned}$$

and

$$\begin{aligned} h^\alpha_{\mu;\nu;\alpha} + h^\alpha_{\nu;\mu;\alpha} &= h^\alpha_{\mu;\alpha;\nu} + h^\alpha_{\nu;\alpha;\mu} \\ &\quad + 2h^{\alpha\sigma} R_{\sigma\mu\nu\alpha} + h^\sigma_{\mu} R_{\sigma\nu} + h^\sigma_{\nu} R_{\sigma\mu} \end{aligned}$$

Thus the perturbed Einstein field equations

assume the alternate form

(10c)

$$-\frac{1}{2} \left[h_{\mu\nu;\alpha}{}^{;\alpha} + 2h^{\alpha\sigma} R_{\sigma\mu\nu\alpha} - h^{\sigma}{}_{\mu} R_{\nu\sigma} - h^{\sigma}{}_{\nu} R_{\mu\sigma} - h^{\alpha}{}_{\mu;\alpha;\nu} - h^{\alpha}{}_{\nu;\alpha;\mu} + h^{\alpha}{}_{\alpha;\mu;\nu} + h_{\mu\nu} R + g_{\mu\nu} \{ h^{\sigma\rho}{}_{;\rho;\sigma} - h^{\alpha}{}_{\alpha;\beta}{}^{;\beta} - h^{\alpha\beta} R_{\alpha\beta} \} \right] = 8\pi \Delta t_m$$

This can be simplified with by introducing the tensor

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h} \quad ; \quad h = h^{\alpha}{}_{\alpha} = \text{trace of } h_{\mu\nu}$$

$$\bar{h} = -h$$

Thus

$$\bar{h}^{\alpha}{}_{\mu;\alpha;\nu} + \bar{h}^{\alpha}{}_{\nu;\alpha;\mu} = h^{\alpha}{}_{\mu;\alpha;\nu} + h^{\alpha}{}_{\nu;\alpha;\mu} - h^{\alpha}{}_{\alpha;\mu;\nu}$$

$$2\bar{h}^{\alpha\sigma} R_{\sigma\mu\nu\alpha} = 2h^{\alpha\sigma} R_{\sigma\mu\nu\alpha} + h R_{\mu\nu}$$

$$\bar{h}^{\sigma}{}_{\mu} R_{\nu\sigma} = h^{\sigma}{}_{\mu} R_{\nu\sigma} - \frac{1}{2} h R_{\mu\nu}$$

also

$$\bar{h}_{\mu\nu} R - g_{\mu\nu} \bar{h}^{\alpha\beta} R_{\alpha\beta} = h_{\mu\nu} R - g_{\mu\nu} h^{\alpha\beta} R_{\alpha\beta}$$

and

$$\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} = h_{\mu\nu;\alpha}{}^{;\alpha} - \frac{1}{2} g_{\mu\nu} h^{\alpha}{}_{\alpha;\beta}{}^{;\beta}$$

$$\bar{h}^{\sigma\rho}{}_{;\rho;\sigma} = h^{\sigma\rho}{}_{;\rho;\sigma} - \frac{1}{2} h^{\alpha}{}_{\alpha;\beta}{}^{;\beta}$$

PLUG IN AND GET

The perturbed Einstein Field Equations

(100)

$$\frac{1}{2} \left[\bar{h}_{\mu\nu;\alpha}{}^{;\alpha} - 2\bar{h}^{\alpha\sigma} R_{\sigma\mu\nu\alpha} - \bar{h}^{\sigma}{}_{\mu} R_{\nu\sigma} - \bar{h}^{\sigma}{}_{\nu} R_{\mu\sigma} - \left(\bar{h}^{\alpha}{}_{\mu;\alpha;\nu} + \bar{h}^{\alpha}{}_{\nu;\alpha;\mu} \right) + \bar{h}_{\mu\nu} R - g_{\mu\nu} \bar{h}^{\alpha\beta} R_{\alpha\beta} \right] = 8\pi G t_{\mu\nu}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^{\alpha}{}_{\alpha}$

The operator that occurs in the 1st line,

$$\Delta_{\mu\nu} = h_{\mu\nu;\alpha}{}^{;\alpha} - h^{\alpha\sigma} R_{\sigma\mu\nu\alpha} - h^{\alpha\sigma} R_{\mu\sigma\alpha\nu} - h^{\sigma}{}_{\nu} R_{\mu\sigma} - h^{\sigma}{}_{\mu} R_{\nu\sigma}$$

is the generalized Laplacian acting on tensors. It has the properties

- (i) it preserves the symmetries of the tensor it acts upon
- (ii) it is self adjoint
- (iii) it commutes with contraction
- (iv) it coincides with the standard Hodge-de Rham Laplacian on p-forms: $\Delta = d\delta + \delta d$.

E: G.W. Gibbons & M.J. Perry, Nucl. Phys. B 146, 90 (1978)

A. Lichnerowicz in Relativity Groups and Topology, Les Houches Lectures 1983

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Errata

1. On P ~~18~~ 18 + 2 the square has been completed incorrectly.
2. On P 18 + 64 replace 8π by 16π in the odd parity field eq'ns.

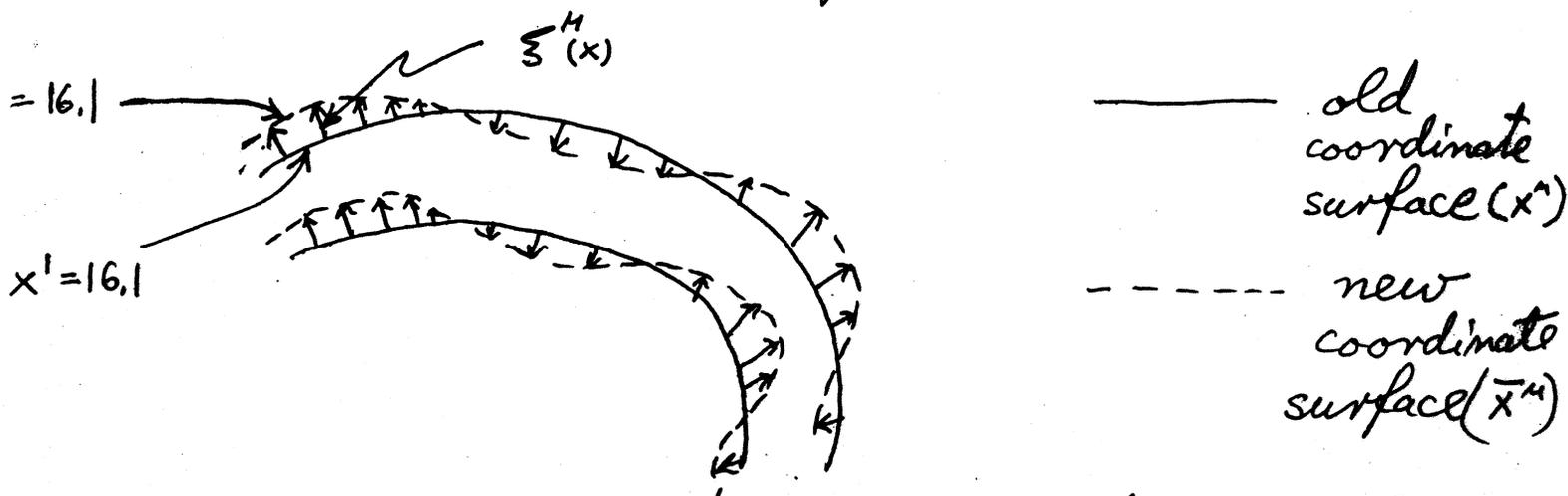
Infinitesimal Coordinate Transformations ⑪ (Perturbations in the background coordinate system.)

Not all perturbations in the metric coefficient functions are due to perturbations in the metric itself. Some perturbations in the coefficients ($g_{\mu\nu}^{(x)} \rightarrow g_{\mu\nu}^{(x)} + \Delta g_{\mu\nu}^{(x)}$) are due to mere perturbational changes in the coordinate system used. From the point of view of describing phenomena involving perturbations in the gravitational field, as well as in other fields, it is clear that changes due to a mere recoordination are of less interest than changes due to actual perturbations in the tensor fields. To see how this works in practice consid

an infinitesimal coordinate change.

$$x^M \rightarrow x^M + \xi^M(x) \equiv \bar{x}^M$$

where $\xi^M(x)$ are the components of an infinitesimal vector field



Evidently, such a coordinate transformation is described by $\xi^M(x)$. The inverse transformation is

$$\begin{aligned} \bar{x}^M \rightarrow x^M &= \bar{x}^M - \xi^M(x) \\ &= \bar{x}^M - \xi^M(\bar{x} - \xi) \\ &= \bar{x}^M - \underbrace{\xi^M(\bar{x})}_{1^{st} \text{ order}} + \underbrace{\xi^M_{,\sigma}(\bar{x}) \xi^\sigma}_{2^{nd} \text{ order}} + \dots \end{aligned}$$

Thus
$$x^M \approx \bar{x}^M - \xi^M(\bar{x})$$

where we neglected 2nd and higher order

terms. Thus we have

$$\begin{array}{|l} \bar{x}^{\mu} = x^{\mu} + \xi^{\mu}(x) \\ \bar{x}^{\mu} = \bar{x}^{\mu} - \xi^{\mu}(\bar{x}) \end{array}$$

Transformation

Inverse transformation

where ξ^{μ} are the components of a vector.

Effect of Infinitesimal Coordinate Transformation on Metric.

The effect is easy to determine. One has

$$\begin{aligned} g_{\mu\nu}(x) dx^{\mu} \otimes dx^{\nu} &= g_{\mu\nu}(\bar{x}^{\sigma} - \xi^{\sigma}(\bar{x})) d(\bar{x}^{\mu} - \xi^{\mu}(\bar{x})) \otimes d(\bar{x}^{\nu} - \xi^{\nu}(\bar{x})) \\ &= g_{\mu\nu}(\bar{x}) d\bar{x}^{\mu} \otimes d\bar{x}^{\nu} - g_{\mu\nu, \sigma} \xi^{\sigma} d\bar{x}^{\mu} \otimes d\bar{x}^{\nu} \\ &\quad - g_{\mu\nu} \xi^{\sigma}_{, \sigma} d\bar{x}^{\mu} \otimes d\bar{x}^{\nu} - g_{\mu\nu} \xi^{\nu}_{, \sigma} d\bar{x}^{\mu} \otimes d\bar{x}^{\sigma} \\ &\quad + 2^{\text{nd}} \text{ and higher order terms} \end{aligned}$$

It follows that the perturbation induced by ξ^{μ} is

$$\begin{aligned} \Delta g_{\mu\nu} &= -g_{\mu\nu, \sigma} \xi^{\sigma} - g_{\sigma\nu} \xi^{\sigma}_{, \mu} - g_{\mu\sigma} \xi^{\sigma}_{, \nu} \\ &= -(\xi_{\mu; \nu} + \xi_{\nu; \mu}) \end{aligned}$$

where comma denotes partial and semicolon covariant derivative

(14)

Problem: verify that

$$g_{\mu\nu,\sigma} \xi^\sigma + g_{\sigma\nu} \xi_{,\mu}^\sigma + g_{\mu\sigma} \xi_{,\nu}^\sigma = \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

Conclusion:

Given an infinitesimal coordinate transformation $\bar{x}^\mu = x^\mu + \xi^\mu(x)$ characterized by the (infinitesimal) vector $\xi^\mu(x) \frac{\partial}{\partial x^\mu}$, then the metric ~~will transform~~ ^{tensor} coefficient field will be transformed as follows:

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x) + h_{\mu\nu}(x)$$

where $h_{\mu\nu}(x) = -(\xi_{\mu;\nu} + \xi_{\nu;\mu})$

Comment:

If $g_{\mu\nu}(x)$ is a solution to the full Einstein field equation, then so is

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$$

But $\bar{g}_{\mu\nu}$ as well as $g_{\mu\nu}$ refer to the same metric. They are different representations

of the same metric.

Comment: Suppose one has two perturbations of the same background (unperturbed) metric, $g_{\mu\nu}$, $h_{\mu\nu}^1$ and $h_{\mu\nu}^2$. Further, suppose that

$$h_{\mu\nu}^2(x) = h_{\mu\nu}^1(x) - \xi_{\mu;\nu}(x) - \xi_{\nu;\mu}(x)$$

Then it is clear that $h_{\mu\nu}^1$ and $h_{\mu\nu}^2$ express the same perturbation of the metric, $g_{\mu\nu}(x)$.

Definition: The transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$$

applied to the perturbation $h_{\mu\nu}$ and induced by the vector field ξ_{μ} is called a gauge transformation.

The metric is not the only tensor field affected by infinitesimal coordinate transformations. There are general tensor fields, say $T_{\mu\nu}(x)$; general covector fields, say $v^{\mu}(x)$; and scalar fields, say $\phi(x)$.

Effect of infinitesimal coordinate transformations on tensor, vector and scalar fields (16)
 Given: $\bar{x}^M = x^M + \xi^M(x)$, an inf.^l coord. transformation.
Tensor:

$$\begin{aligned}
 t_{\mu\nu}(x) dx^\mu \otimes dx^\nu &= t_{\mu\nu}(\bar{x} - \xi) d(\bar{x}^\mu - \xi^\mu) \otimes d(\bar{x}^\nu - \xi^\nu) = \\
 &= t_{\mu\nu}(\bar{x}) d\bar{x}^\mu \otimes d\bar{x}^\nu - t_{\mu\nu,\sigma} \xi^\sigma d\bar{x}^\mu \otimes d\bar{x}^\nu \\
 &\quad - t_{\mu\nu} \xi^\mu_{,\sigma} d\bar{x}^\sigma \otimes d\bar{x}^\nu - t_{\mu\nu} \xi^\nu_{,\sigma} d\bar{x}^\mu \otimes d\bar{x}^\sigma \\
 &\quad + \text{higher order terms.}
 \end{aligned}$$

Thus

$$\Delta t_{\mu\nu} = - t_{\mu\nu,\sigma} \xi^\sigma - t_{\sigma\nu} \xi^\sigma_{,\mu} - t_{\mu\sigma} \xi^\sigma_{,\nu}$$

$$\Delta t_{\mu\nu} = - t_{\mu\nu;\sigma} \xi^\sigma - t_{\sigma\nu} \xi^\sigma_{;\mu} - t_{\mu\sigma} \xi^\sigma_{;\nu}$$

Problem: show that the above equality does, in fact, hold true.

covector:

$$\begin{aligned}
 v_\mu dx^\mu &= v_\mu(\bar{x} - \xi) d(\bar{x}^\mu - \xi^\mu) \\
 &= v_\mu(\bar{x}) d\bar{x}^\mu - v_{\mu,\sigma} \xi^\sigma d\bar{x}^\mu - v_\mu \xi^\mu_{,\sigma} dx^\sigma
 \end{aligned}$$

Thus

$$\Delta v_\mu = - v_{\mu,\sigma} \xi^\sigma - v_\sigma \xi^\sigma_{,\mu}$$

$$\Delta v_\mu = - (v_{\mu;\sigma} \xi^\sigma + v_\sigma \xi^\sigma_{;\mu})$$

Problem: show that the above equality does in fact, hold true

vector:

$$\begin{aligned} u^\mu \frac{\partial}{\partial x^\mu} &= u^\mu (\bar{x}^\nu - \xi^\nu) \frac{\partial \bar{x}^\sigma}{\partial x^\mu} \frac{\partial}{\partial \bar{x}^\sigma} \\ &= u^\mu (\bar{x}^\nu - \xi^\nu) (\delta_{\mu}^{\sigma} + \xi_{,\mu}^{\sigma}) \frac{\partial}{\partial \bar{x}^\sigma} \\ &= u^\mu \frac{\partial}{\partial \bar{x}^\mu} - u^\mu_{,\nu} \xi^\nu \frac{\partial}{\partial \bar{x}^\mu} + u^\mu \xi_{,\mu}^{\sigma} \frac{\partial}{\partial \bar{x}^\sigma} \end{aligned}$$

thus

$$\Delta u^\mu = - (u^\mu_{,\sigma} \xi^\sigma - u^\sigma \xi_{,\sigma}^\mu)$$

THIS WILL BE RECOGNIZED AS THE COMPONENTS OF $-\llbracket u, \xi \rrbracket$, the commutator of u and ξ !

$$\Delta u^\mu = - (u^\mu_{;\sigma} \xi^\sigma - u^\sigma \xi_{;\sigma}^\mu)$$

Problem: show that the equality

$$u^\mu_{,\sigma} \xi^\sigma - u^\sigma \xi_{,\sigma}^\mu = u^\mu_{;\sigma} \xi^\sigma - u^\sigma \xi_{;\sigma}^\mu$$

does in fact hold true.

scalar:

$$\begin{aligned} \phi(x^\mu) &= \phi(\bar{x}^\mu - \xi^\mu) \\ &= \phi(\bar{x}^\mu) - \phi_{,\sigma} \xi^\sigma \end{aligned}$$

thus

$$\Delta \phi = - \phi_{,\sigma} \xi^\sigma$$

Thus the change here is merely the directional derivative.

Definition: Consider perturbations

$$\Delta h_{\mu\nu}, \Delta t_{\mu\nu}, \Delta v_{\mu}, \Delta u^{\mu}, \Delta \phi$$

in the geometrical objects

$$g_{\mu\nu}, t_{\mu\nu}, v_{\mu}, u^{\mu}, \phi.$$

Let ξ^{μ} induce an infinitesimal coordinate transformation in these background geometrical objects. The following transformations are called gauge transformations.

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu})$$

$$\Delta t_{\mu\nu} \rightarrow \Delta t_{\mu\nu} - (t_{\mu\nu;\sigma} \xi^{\sigma} + t_{\sigma\nu} \xi^{\sigma}_{;\mu} + t_{\mu\sigma} \xi^{\sigma}_{;\nu})$$

$$\Delta v_{\mu} \rightarrow \Delta v_{\mu} - (v_{\mu;\sigma} \xi^{\sigma} + v_{\sigma} \xi^{\sigma}_{;\mu})$$

$$\Delta u^{\mu} \rightarrow \Delta u^{\mu} - (u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma})$$

$$\Delta \phi \rightarrow \Delta \phi - (\phi_{;\sigma} \xi^{\sigma})$$

Comment: The expressions in the parenthesis on the r.h.s., "the gauge changes", are the Lie derivatives of the geometrical objects.

Notation:
 Lie derivative w.r.t. ξ^{μ} of $t_{\mu\nu}$:

$$L_{\xi} t_{\mu\nu} = t_{\mu\nu;\sigma} \xi^{\sigma} + t_{\sigma\nu} \xi^{\sigma}_{;\mu} + t_{\mu\sigma} \xi^{\sigma}_{;\nu}$$

similarly for others

Generic Spherically Symmetric Space

①

times.

I) THE METRIC

Consider the metric for an arbitrary spherically symmetric space time

$$g_{\mu\nu} dx^\mu dx^\nu = g_{AB}(x^C) dx^A dx^B + r^2(x^C) [d\theta^2 + \sin^2\theta d\phi^2]$$
$$= g_{00} dx^0 dx^0 + 2g_{01} dx^0 dx^1 + g_{11} dx^1 dx^1 + r^2 [d\theta^2 + \sin^2\theta d\phi^2]$$

Thus one has the metric

$$g_{ab} dx^a dx^b = r^2(x^C) [d\theta^2 + \sin^2\theta d\phi^2]$$

on the concentric spheres of radius $r(x^C)$, and the metric

$$g_{AB} dx^A dx^B$$

on the space-time plane perpendicular to these spheres.

~~There exists~~

Theorem: There exists a coordinate

system (T, R) such that the metric has diagonal form with respect to the corresponding coordinate basis, i.e.

~~Theorem~~ : (cont'd)

the metric can be brought into the form (2)

$$g_{\mu\nu} dx^\mu dx^\nu = a(T, R) dT^2 + b(T, R) dR^2 + r^2 [d\theta^2 + \sin^2\theta d\phi^2]$$

Proof: (1) consider

$$\begin{aligned} & g_{00}(x^0, x')(dx^0)^2 + 2g_{01}(x^0, x') dx^0 dx^1 + g_{11}(x^0, x')(dx^1)^2 \\ &= \left(g_{00} dx^0 + \frac{g_{01}}{g_{00}} dx^1 \right)^2 + \left(g_{11} - \frac{g_{01}^2}{g_{00}} \right) dx^1{}^2 \end{aligned}$$

by completing the square

2) Consider the differential equation

$$g_{00}(x^0, x') dx^0 + \frac{g_{01}(x^0, x')}{g_{00}} dx^1 = 0$$

in x^0 and x^1 . This equation can always be solved (provided g_{00} and $\frac{g_{01}}{g_{00}}$ satisfy suitable continuity conditions):

$$x^0 = f(x^1, T) \quad (*)$$

where T is an integration constant. Solve Eq. (*) for T :

$$T = g(x^0, x') \quad (**)$$

③ The differential of $(**)$,

$$dT = \frac{\partial g}{\partial x^0} dx^0 + \frac{\partial g}{\partial x^1} dx^1,$$

if one evaluates it along the tangent direction tangent to the world lines $(**)$, i.e.

$$\Delta T = 0 = \frac{\partial g}{\partial x^0} \Delta x^0 + \frac{\partial g}{\partial x^1} \Delta x^1$$

The given differential equation reads

$$0 = g_{00} \Delta x^0 + \frac{g_{01}}{g_{00}} \Delta x^1$$

It follows that the vectors $(g_{00}, \frac{g_{01}}{g_{00}})$

and $(\frac{\partial g}{\partial x^0}, \frac{\partial g}{\partial x^1})$ in the two plane are codirectional, i.e. \exists a scalar $\lambda(x^0, x^1)$ function ("integrating factor") such that

$$dT = dg = \lambda (g_{00} dx^0 + \frac{g_{01}}{g_{00}} dx^1) \text{ is exact.}$$

④ Thus the metric becomes

$$g_{AB} dx^A dx^B = \underbrace{\frac{1}{\lambda^2}}_{a(T,R)} dT^2 + \underbrace{\left(g_{11} - \frac{g_{01}^2}{g_{00}}\right)}_{b(T,R)} dx'^2.$$

end of proof.

Summary:

(4)

$$(x^0; x^1) \longrightarrow (T, R) = (q(x^0; x^1); x^1)$$

bring metric coefficients into diagonal form.

$$g_{AB} dx^A dx^B \longrightarrow a(T, R) dT^2 + b(T, R) dR^2.$$

Conclusion: ~~The~~ For computational purposes (for example) one may write the metric on a generic spherically symmetric space time in the form

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi(R, T)} dT^2 + e^{2\Lambda(R, T)} dR^2 + r^2(R, T) [d\theta^2 + \sin^2\theta d\phi^2]$$

II Curvature Tensor.

(5)

Comments:

1. It is evident that there is a natural metric tensor field

$$g_{AB} dx^A dx^B$$

defined on the two dimensional space time spanned by the time and radial coordinates x^0 and x^1 . Call this manifold M^2 ,

2. The metric coefficient $r^2(x^0)$ is a scalar on M^2 . (hence $r(x^0)$)

3. $v_A = \left(\frac{\dot{r}}{r}, \frac{r'}{r} \right)$ are the components of the gradient (covector) on M^2 .

4. Using $g_{AB} dx^A dx^B$ one may define parallel transport, and hence the ~~the~~ Christoffel symbols on M^2

$$\Gamma_{BC}^A = \frac{1}{2} g^{AD} (g_{DB,C} + g_{DC,B} - g_{BC,D})$$

5. We can define the covariant derivative of v_A on M^2 : (next page)

$$\nu_{A|B} = \nu_{A,B} - \nu_c \Gamma_{AB}^c \quad (6)$$

One can compute the component of the Riemann Christoffel curvature tensor on the four-dimensional space time. The result can be expressed in terms of geometrical objects (tensors, vectors, scalars) on M^2 .

Starting with the metric in diagonal form, introducing $\nu_A, \nu_{A|B}, r^2$ as well as the Gaussian curvature $R = \frac{1}{2}$ curvature invariant R^c_c on M^2 one obtains the following results

$$R_{AaBb} = -(\nu_{A|B} + \nu_A \nu_B)^{(2)} g_{ab}$$

$$R^a_{bcd} = \left(\frac{1}{r^2} - \nu_A \nu^A \right) \left[\delta^a_c g_{bd} - \delta^a_d g_{bc} \right]$$

$$R^A_{BCD} = R \left[\delta^A_C g_{BD} - \delta^A_D g_{BC} \right]$$

where $\left. \begin{matrix} A \\ B \\ C \\ D \end{matrix} \right\} = 0, 1 \text{ only}$ $\left. \begin{matrix} a \\ b \\ c \\ d \end{matrix} \right\} = \theta, \varphi \text{ only}$ (5)

c) The components of the Ricci tensor $R^\sigma_{\mu\nu}$ are

$$\begin{aligned} R_{AB} &= R^C_{ACB} + R^C_{A c B} \\ &= \cancel{R} g_{AB} - 2(v_{A|B} + v_A v_B) \end{aligned}$$

$$R_{Aa} = 0$$

$$R_{ab} = R^c_{acb} + R^c_{a c b}$$

$$= - (v_c^{1c} + v_c v^c)^{(2)} g_{ab} + \left(\frac{1}{r^2} - v_c v^c\right)^{(2)} g_{ab}$$

$$= - \left(v_c^{1c} + 2v_c v^c - \frac{1}{r^2}\right) g_{ab}$$

d) Curvature invariant for 4-dim. space time

$$R = R^A_A + R^a_a$$

$$= 2R - 2(v_c^{1c} + v_c v^c) - 2(v_c^{1c} + 2v_c v^c - \frac{1}{r^2})$$

$$= 2R - 2(2v_c^{1c} + 3v_c v^c - \frac{1}{r^2})$$

E) Einstein tensor & field equations (8)

$$G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R$$

$$G_{AB} = -2(\nu_{A|B} + \nu_{A\nu} \nu_B) + g_{AB} (2\nu_c{}^{1c} + 3\nu_c \nu^c - \frac{1}{r^2})$$

$$G_{Aa} = 0 \quad \text{because} \quad g_{Aa} = 0$$

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

$$G_{ab} = \left\{ (\nu_c{}^{1c} + \nu_c \nu^c) - R \right\} g_{ab}$$

The stress energy tensor on a spherically symmetric time has the form

$$T_{\mu\nu} dx^\mu dx^\nu = t_{AB} dx^A dx^B + \frac{1}{2} t_c g_{ab} dx^a dx^b$$

∴ Field eq'ns for generic space time are

$$-2(\nu_{A|B} + \nu_A \nu_B) + g_{AB} (2\nu_c{}^{1c} + 3\nu_c \nu^c - \frac{1}{r^2}) = 8\pi T_{AB}$$

$\nu_c{}^{1c} + \nu_c \nu^c - R = 8\pi \frac{1}{2} t_a{}^a$
which are a tensor and a scalar equation on \mathbb{R}^4 .

Problem

(86)

Derive the Einstein field equations

$$8\pi T_{AB} = -2(\psi_{A|B} + \psi_A \psi_B) + g_{AB} (2\psi_c{}^c + 3\psi_c \psi^c - \frac{1}{r^2})$$

$$8\pi T_a{}^a = 2[\psi_c{}^c + \psi_c \psi^c - R]$$

from a variational principle.

Comment: This is a publishable result.

First attempt: use $\int \int \int \int ({}^{(4)}R \sqrt{-g}) d^4x$ as the starting point. a) $({}^{(4)}R = R^A{}_A + R^a{}_a$

$$= 2R - 2(2\psi_c{}^c + 3\psi_c \psi^c - \frac{1}{r^2})$$

$$b) \sqrt{-g} = \sqrt{-g} r^2 \sin\theta$$

$$c) \int \int \int \int ({}^{(4)}R \sqrt{-g}) d^4x = 4\pi \cdot 2 \int \int (R - 2\psi_c{}^c + 3\psi_c \psi^c - \frac{1}{r^2}) r^2 \sqrt{g} dx^i$$

Thus the reduced ^{Action} Lagrangian integral is

$$8\pi \int \int [r^2 \sqrt{g} R - 2r^2 \sqrt{g} \psi_c{}^c + 3r^2 \sqrt{g} \psi_c \psi^c - \sqrt{g}] dx$$

The variations trial functions are $r(x, x')$ and $g_{AB}(x, x')$ and their variations δr and δg_{AB} presumably yield a change in the action which is

$$8\pi \int \int \left\{ (?) \delta r + (?) \delta g_{AB} + \text{pure divergence} \right\} dx$$

Scalar, Vector and Tensor Harmonics ⑨

on the unit two sphere.

Our subject of study is some arbitrarily given spherically symmetric space time characterized by the metric

$$ds^2 = g_{AB}(x^C) dx^A dx^B + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

The assumed arbitrariness manifests itself in ~~Although the~~ the fact that we do not specify $g_{AB} dx^A dx^B$.

However, the rest of the metric is quite well defined, once r^2 is given. The rest is therefore essentially determined by

$$r_{ab} dx^a dx^b = d\theta^2 + \sin^2 \theta d\phi^2$$

the metric on the unit two sphere, S^2 .

Since we wish to study scalar, vector and tensor fields (e.g. Klein Gordon fields, electro-magnetic fields, and gravitational fields wave)

on space time, we must at a minimum become very familiar with scalar, vector and tensor fields on S^2 , the unit two sphere.

* This we do by finding ^{or monormal} scalar, vector, and (10) tensor basis harmonics (on the two sphere) which in term of any suitably well behaved scalar, vector and tensor harmonic can be expanded.

I. Scalar Harmonics on S^2 .

These harmonics are defined by the following two properties

(i) Let f be a homogeneous polynomial of degree n , and let f be defined on 3-dimensional cartesian space coordinate by the coordinates x^1, x^2, x^3 :

$$f(x^1, x^2, x^3) = r^n f\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right)$$

NOTE: if $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ then

$f\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right) = Y_n(\theta, \varphi)$ is actually a function defined on the unit two sphere, which is coordinatized by θ and φ .

(ii) Let f be a harmonic polynomial, i.e. (11)

$$0 = \nabla^2 f = \frac{\partial^2 f}{\partial x^1{}^2} + \frac{\partial^2 f}{\partial x^2{}^2} + \frac{\partial^2 f}{\partial x^3{}^2}$$

NOTE: Typically f has then the form

$$f = \sum_{\substack{i, j, \dots, k=1 \\ i, j, \dots, k=1}}^3 A_{i, j, \dots, k} \underbrace{x^i x^j \dots x^k}_{n \text{ factors}}$$

where $A_{i, j, \dots, k}$ are certain constants.

II Properties of scalar harmonics.

a) transformation under parity operation

Definition:

$$(x^1, x^2, x^3) \xrightarrow{P} (-x^1, -x^2, -x^3)$$

$$(\theta, \varphi) \xrightarrow[\sigma]{P} (\pi - \theta, \varphi + \pi) \text{ on } S^2$$

is called the parity transformation

It is clear that

$$P f\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right) = f\left(\frac{-x^1}{r}, \frac{-x^2}{r}, \frac{-x^3}{r}\right) = (-1)^n f\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right)$$

$$\text{or } \boxed{P Y_n(\theta, \varphi) = (-1)^n Y_n(\theta, \varphi)}$$

$Y_n(\theta, \varphi)$ is said to be of parity $(-1)^n$. Regge and Wheeler, Phys. Rev. 108, 1063 (1957) call this even parity

b) Eigenvalue equation satisfied by $Y_n(\theta, \phi)$

$f = r^n Y_n(\theta, \phi)$ is a harmonic polynomial and it satisfies therefore

$$0 = \nabla^2 f = g^{ij} f_{,ij} \quad \text{in flat three space}$$

whose metric in spherical coordinates has the form

$$(i) \quad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ \equiv dr^2 + r^2 \gamma_{ab} dx^a dx^b$$

$$\therefore g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 \gamma_{ab} \\ 0 & 0 & 0 \end{bmatrix}$$

$\left. \begin{matrix} a \\ b \end{matrix} \right\} = \left\{ \begin{matrix} \theta \\ \phi \end{matrix} \right.$

ii) the Christoffel symbols Γ_{ij}^k are: $\left. \begin{matrix} i \\ j \\ k \end{matrix} \right\} = 1, 2, 3$

$$\left. \begin{matrix} \Gamma_{\theta\theta}^r = -r \\ \Gamma_{\phi\phi}^r = -r \sin^2 \theta \end{matrix} \right\} \Gamma_{ab}^r = -\frac{1}{r} g_{ab} = -r \gamma_{ab} \quad \text{metric on } S^2$$

$$\left. \begin{matrix} \Gamma_{r\theta}^\theta = \frac{1}{r} \\ \Gamma_{r\phi}^\phi = \frac{1}{r} \end{matrix} \right\} \Gamma_{r b}^a = \frac{1}{r} \delta_{ab}^a \quad \text{on } S^2$$

$$\left. \begin{matrix} \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta \\ \Gamma_{\phi\theta}^\phi = \cot \theta \end{matrix} \right\} \Gamma_{bc}^a \quad \text{Christoffel symbols on } S^2$$

All other Christoffel symbols are zero.

(iii) Now rewrite $\nabla^2 f$ in terms of covariant derivatives on S^2 : (13)

$$0 = g^{ij} f_{,i;j} = f_{,r;r} + f_{,a;b} g^{ab}$$

$$= f_{,r;r} - f_{,k} \Gamma_{rr}^k + g^{ab} (f_{,a;b} - f_{,k} \Gamma_{ab}^k)$$

where $k = 1, 2, 3$

$$= f_{,r;r} - f_{,r} \underbrace{\Gamma_{rr}^r}_{\text{zero}} - f_{,c} \underbrace{\Gamma_{rr}^c}_{\text{zero}}$$

$$+ g^{ab} (f_{,a;b} - f_{,r} \Gamma_{ab}^r - f_{,c} \Gamma_{ab}^c)$$

$$= f_{,r;r} - 0 - 0$$

$$g^{ab} (f_{,a;b} - f_{,r} \Gamma_{ab}^r)$$

where ∇ denotes covariant derivative on S^2 .

$$= f_{,r;r} + \frac{2}{r} f_{,r} + f_{,a;b} \frac{\gamma^{ab}}{r^2}$$

Note: derivatives (covariant or otherwise) with respect to θ and ϕ (i.e. a or b) do not affect r

Result: use $f = r^n \chi_n(\theta, \phi)$ to obtain

$$\chi_{n,a;b} \gamma^{ab} = -n(n+1) \chi_n$$

Eigenvalue eq'n on S^2 .

III Vector Harmonics on S^2

(17)

It is clear that the scalar harmonics ^{just} constructed can be used to obtain ^{the} vector harmonics

$$Y_{n,a} dx^a = Y_{n,\theta} d\theta + Y_{n,\phi} d\phi$$

Besides these scalarly derived vector harmonics there is also a set of harmonics that can not be derived from a scalar.

Its properties are three-fold

(i) Consider a covector field (on S^2) which has no radial component

$$P_1 dx^1 + P_2 dx^2 + P_3 dx^3 = \underbrace{P_r}_{\text{zero}} dr + P_\theta d\theta + P_\phi d\phi$$

$$\boxed{P_r = 0}$$

(ii) The ~~exp~~ covector components w.r.t to the o.n. basis $\{dx^1, dx^2, dx^3\}$ are n^{th} degree homogeneous polynomials

$$\text{Thus } P_i dx^i = B_{i,j,\dots,k} x^j \dots x^k dx^i$$

where
$$\mathcal{Y}_i(x^1, x^2, x^3) = r^n \mathcal{S}_i\left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right) \equiv r^n S_i(\theta, \varphi)$$
(15)

} component
w.r.t,
orthonor-
mal basis.

(iii) The covector field is harmonic

i.e

$$0 = \nabla^2 \mathcal{Y}_i = \mathcal{Y}_{i;j}{}^{;j} \quad \left. \begin{matrix} i \\ j \end{matrix} \right\} = 1, 2, 3$$

IV Properties of these vector harmonics.

● a) Eigenvalue equation

compute $\nabla^2 \mathcal{Y}_r$, $\nabla^2 \mathcal{Y}_\theta$, $\nabla^2 \mathcal{Y}_\varphi$ and get for Laplace's eq'n in spherical coordinates:

$$0 = \nabla^2 \mathcal{Y}_i$$

(i) for $i = a$ ($= \theta$ or φ) we get

$$\nabla^2 \mathcal{Y}_a = \mathcal{Y}_{a;r,r} - \frac{1}{r^2} \mathcal{Y}_a + \frac{1}{r^2} \mathcal{Y}_{a|b}{}^{|b} = 0$$

where $|$ denotes covariant derivative on S^2 .

● and where we used $\mathcal{Y}_r = 0$

(ii) for $i = r$ we get

$$\nabla^2 \mathcal{L}_r = \mathcal{L}_{r; i; j} g^{ij} = \mathcal{L}_{r; a; b} g^{ab} = -\frac{2}{r^3} \mathcal{L}_{ab} \gamma^{ab}$$

16

The components \mathcal{L}_a in (i) and (ii) are coordinate components,
i.e. (*) $\mathcal{L}_2 dx^i = \mathcal{L}_r dr + \mathcal{L}_\theta d\theta + \mathcal{L}_\phi d\phi$
and we need the angular harmonics on S^2 . Thus

$$\mathcal{L}_2 (x^{\hat{1}}, x^{\hat{2}}, x^{\hat{3}}) dx^i = r^n \mathcal{L}_2 \left(\frac{x^{\hat{1}}}{r}, \frac{x^{\hat{2}}}{r}, \frac{x^{\hat{3}}}{r} \right) dx^i$$

hats indicate
o.n. basis
3 dim. space

$$(1) \quad = r^n S_{\hat{2}}(\theta, \phi) dx^i$$

$$(2) \quad = r^n \left[S_{\hat{\theta}}(\theta, \phi) r d\theta + S_{\hat{\phi}}(\theta, \phi) r \sin\theta d\phi \right]$$

$$(3) \quad = r^{n+1} \left[S_{\hat{\theta}}(\theta, \phi) d\theta + S_{\hat{\phi}}(\theta, \phi) \sin\theta d\phi \right]$$

$$(4) \quad = r^{n+1} \left[S_a(\theta, \phi) dx^a \right]$$

An objection may be raised as to what precisely
is the basis in expression (3) as compared to (4).

In (3), $S_{\hat{\theta}} d\theta + S_{\hat{\phi}} \sin\theta d\phi$

seems to indicate that the basis is ~~the~~ o.n.
w.r.t the metric

$$d\theta \otimes d\theta + (\sin\theta d\phi) \otimes (\sin\theta d\phi).$$

By contrast, ~~$S_{\hat{\theta}} r d\theta$~~ in (2), $S_{\hat{\theta}} r d\theta + S_{\hat{\phi}} r \sin\theta d\phi$

seems to indicate that the basis is ~~not~~ O.N. ~~basis~~ (17)
 w.r.t. the metric

$$(r d\theta) \otimes (r d\theta) + (r \sin\theta d\varphi) \otimes (r \sin\theta d\varphi)$$

The contradiction is resolved by the observation that on the unit sphere (and only on the unit sphere) the two metrics coincide. ~~so that~~ thus the results of computing lengths, writing down vector field components w.r.t. an O.N. normalized basis ^{etc} even though done using different metrics, coincide. This coincidence does not hold if one goes away from the unit sphere.

Comparing coefficients in expression (4) with those in (*) one has

$$\mathcal{L}_a(x^i) = r^{n+1} S_a(\theta, \varphi) \quad a = (\theta, \varphi)$$

$$\mathcal{L}_r(x^i) = 0 \quad \text{already given.}$$

Substitute these expressions into $\nabla^2 \mathcal{L}_i = 0$ in (i) and (ii) yields

$$\begin{aligned} S_{a|b}{}^b &= -(n^2 + n - 1) S_a \\ S_a{}^{|a} &= 0 \end{aligned}$$

Eigenvalue eq'n for transverse vector harmonic on S^2 .

b) Transformation properties under the ⁽¹⁸⁾ parity operation defined on $P 11$.

$$P: x^1 x^2 x^3 \rightarrow (-x^1, -x^2, x^3)$$

$$\text{or } (\theta, \varphi) \rightarrow (\pi - \theta, \pi + \varphi)$$

We have now two covector fields on S^2 :

$$\cancel{Y_{n,a}(\theta,\varphi) dx^a} = Y_{n,\theta}(\theta,\varphi) d\theta + Y_{n,\varphi}(\theta,\varphi) d\varphi \quad \text{"scalarly derived"}$$

and

$$S_{na} dx^a = S_{n\theta}(\theta,\varphi) d\theta + S_{n\varphi}(\theta,\varphi) d\varphi \quad \text{"transverse" } (S_a{}^a = 0)$$

where $S_{na} dx^a$ is obtained from

$$\int_i dx^i = B_{ij\dots k} x^j \dots x^k dx^i = r^{n+1} S_a(\theta,\varphi) dx^a$$

① Transformation properties of scalarly derived vector harmonic

$$P Y_{n,a} dx^a = \frac{\partial Y_n(\pi-\theta)}{\partial(\pi-\theta)} d(\pi-\theta) + \frac{\partial Y_n(\pi-\theta, \pi+\varphi)}{\partial(\pi+\varphi)} d(\pi+\varphi)$$

$$= (-1)^n \left[\frac{\partial Y_n(\theta,\varphi)}{\partial\theta} d\theta + \frac{\partial Y_n(\theta,\varphi)}{\partial\varphi} d\varphi \right]$$

$$P Y_{n,a} dx^a = (-1)^n Y_{n,a} dx^a$$

"even" parity

② Transformation properties of transverse harmonics $S_a(\theta, \varphi) dx^a$: (19)

$$P \int_i dx^i = \cancel{(-1)^{n+1}} P B_{ij \dots k} x^j \dots x^k dx^i = (-1)^{n+1} \int_i dx^i$$

Thus

$$\boxed{P S_a dx^a = (-1)^{n+1} S_a dx^a}$$

"odd" parity $= (-1)^{n+1} S_a dx^a$

Final result

$$P : Y_{n,a} dx^a \rightarrow (-1)^n Y_{n,a} dx^a$$

$$P : S_{n,a} dx^a \rightarrow (-1)^{n+1} S_{n,a} dx^a$$

i.e. the transverse vector harmonic of degree n has opposite parity of the scalarly derived vector harmonic of degree n .

We shall only consider symmetric rank 2 tensor fields. Tensor harmonics on S^2 are derived from the scalar $Y(\theta, \varphi)$ and from the transverse vector harmonic $S_a(\theta, \varphi)$. For harmonics characterized by the integer n we have

$$Y_{,a;b} dx^a \otimes dx^b \text{ whose parity is } (-1)^n \quad (1)$$

$$Y \gamma_{ab} dx^a \otimes dx^b \text{ whose parity is } (-1)^n \quad (2)$$

$$(S_{a;b} + S_{b;a}) dx^a \otimes dx^b \text{ whose parity is } (-1)^{n+1} \quad (3)$$

Here the colon ":" indicates covariant derivative on S^2 and $\gamma_{ab} dx^a \otimes dx^b = d\theta^2 + \sin^2 d\varphi^2$ on S^2 .

An alternative, but equivalent set of tensor harmonics of degree n is

$$[Y_{,a;b} - n(n+1)Y\gamma_{ab}] dx^a \otimes dx^b \text{ (traceless)} \quad (4)$$

$$Y \gamma_{ab} dx^a \otimes dx^b \quad (5)$$

$$(S_{a;b} + S_{b;a}) dx^a \otimes dx^b \text{ (traceless)} \quad (6)$$

Both $\{(1), (2), (3)\}$ and $\{(4), (5), (6)\}$ are linearly independent sets, but (4), (5), and (6) are also mutually orthogonal.

V Orthogonality Properties

We have constructed the following harmonics of degree n on the unit two sphere S^2

	PARITY
Scalar harmonic: $Y(\theta, \varphi)$	$(-1)^n$ "even"
Vector harmonics: $Y_{,a}^n dx^a$ $S_a^n dx^a$ where $S_a^{n;a} = 0$	$(-1)^n$ "even" $(-1)^{n+1}$ "odd"
Tensor harmonics: $Y_{ab}^n dx^a \otimes dx^b$ $[Y_{,a;b}^n + n(n+1)Y_{ab}^n] dx^a \otimes dx^b$ $(S_{a;b}^n + S_{b;a}^n) dx^a \otimes dx^b$	$(-1)^n$ "even" $(-1)^n$ "even" $(-1)^{n+1}$ "odd"

These harmonics are orthogonal in the following sense

Scalar: $\iint_{S^2} Y^n Y^{n'} d(\text{area}) = 0$ if $n \neq n'$

Vector: $\iint_{S^2} Y_{,a}^n Y_{,b}^{n'} \gamma^{ab} d(\text{area}) = 0 \quad n \neq n' \quad (22)$

$\iint_{S^2} S_a^n S_b^{n'} \gamma^{ab} d(\text{area}) = 0 \quad n \neq n'$

Tensor: $\iint_{S^2} Y_{ab}^n Y_{cd}^{n'} \gamma^{ac} \gamma^{bd} d(\text{area}) = 0 \quad n \neq n'$

$\iint_{S^2} Y_{,a;b}^n Y_{,c;d}^{n'} \gamma^{ac} \gamma^{bd} d(\text{area}) = 0 \quad n \neq n'$

$\iint_{S^2} (S_{a;b}^n + S_{b;a}^n) (S_{c;d}^{n'} + S_{d;c}^{n'}) \gamma^{ac} \gamma^{bd} d(\text{area}) = 0 \quad n \neq n'$

where $d(\text{area}) = \sin\theta d\theta d\phi$

and the integration limits span the whole

sphere: $\theta: 0 \rightarrow \pi$
 $\phi: 0 \rightarrow 2\pi$

These above identities are proved by

using $Y_{,a;b}^n \gamma^{ab} = -(n+1)n Y^n$

$S_{a;b;c}^n \gamma^{bc} = -(n^2+n-1) S_a^n$

and doing several integrations by parts
 wherever necessary.

The following fact is very important: (23)

Theorem: An even parity harmonic is orthogonal to any odd parity harmonic

$$\text{Thus: } \iint_{S^2} Y_{,a} S_b \gamma^{ab} d(\text{area}) = 0$$

$$\iint Y_{,a;b} (S_{c;d} + S_{d;c}) \gamma^{ac} \gamma^{bd} d(\text{area}) = 0$$

$$\iint Y \gamma_{ab} (S^{a;b} + S^{b;a}) d(\text{area}) = 0$$

Proof: ~~Use~~ ~~$S_a{}^{;a} = 0$~~
Do some integration by parts and use

$$S_a{}^{;a} = 0$$

Comment: We also have for the even tensor harmonics

$$\iint Y_{,ab} \left[Y_{,a;b} + n(n+1) \gamma^{ab} \right] d(\text{area}) = 0$$

i.e. the traceless ~~even~~ even harmonic is always orthogonal to $Y \gamma_{ab}$

Summary: The scalar, vector, and tensor harmonics ~~on~~ on S^2 (see P 21) form an orthogonal and hence linearly independent set mutually.

VI ^{characterization} Completeness of harmonics on S^2

(24)

The properties of harmonics on S^2 arise from the eigenvalue equations

$$Y_{,a}^n = -n(n+1) Y^n(\theta, \varphi)$$

$$S_{a;b}^n = -(n^2 + n - 1) S_a^n(\theta, \varphi)$$

Upon solving these explicitly one finds that

$$Y^n(\theta, \varphi) = (a \text{ fn of } \theta)^n \cdot e^{im\varphi} \quad (1)$$

$$S_a^n(\theta, \varphi) = (a \text{ fn of } \theta)_a^n \cdot e^{im\varphi} \quad (2)$$

where m is an integer. ~~The~~ Furthermore, in order that first factor in Eqs (1) and (2) be continuous one must have only

$$|m| \leq n$$

Thus the solutions to the eigenvalue equations

$$Y_{,a}^l = -l(l+1) Y^l$$

$$S_{a;b}^l = -(l^2 + l - 1) S_a^l$$

are in fact characterized by two integers l and m (2)

Here, and from now on, in order to conform with standard notation in book and journals we use the letter l instead of n to denote the degree of a harmonic)

Type of harmonic	Symbol	allowed integers	Parity
Scalar	$Y_{(\theta, \varphi)}^{lm}$	$l = 0, 1, 2, 3$ $-l \leq m \leq l$	$(-1)^l$
Vector	$Y_{,a}^{lm}$	$l = 1, 2, 3$ $-l \leq m \leq l$	$(-1)^l$
	NOTE: $Y_{,a}^{lm}$ vanishes identically for $l=0$		
Vector	S_a^{lm}	$l = 1, 2, 3$ $-l \leq m \leq l$	$(-1)^{l+1}$
	NOTE: S_a^{lm} vanishes identically for $l=0$		
Tensor	Y_{ab}^{lm}	$l = 0, 1, 2, 3$ $-l \leq m \leq l$	$(-1)^l$
Tensor	$Y_{,a;b} + l(l+1) Y_{ab}$	$l = 2, 3, 4, \dots$ $-l \leq m \leq l$	$(-1)^l$
NOTE: This harmonic vanishes identically for $l=0$ and $l=1$			

Tensor

$$S_{aib}^{lm} + S_{bia}^{lm}$$

$$l = 2, 3, 4, \dots$$

$$-l \leq m \leq l$$

$$(-1)^{l+1} \quad (26) \quad (\text{odd})$$

NOTE: The odd parity tensor harmonic vanishes identically for $l = 0$ and $l = 1$

VII Some Useful Identities Obeyed by Harmonics on S^2 ; (suppress explicit reference to integers l & m)

EVEN:

$$Y_{,a}^{:a} = -l(l+1) Y$$

$$Y_{,a;b}^{:b} = - [\quad] Y_{,a}$$

$$Y_{,a;b;c}^{:c} = - [\quad] Y_{,a;b} + [\quad] Y_{,c}$$

$$(Y_{ab})^{:c} = - [\quad] Y_{ab}$$

$$(Y_{,a;b} + l(l+1) Y_{ab})^{:c} = - [\quad] Y_{,a;b} + [\quad] Y_{,c}$$

ODD:

$$S_a^{:a} = 0$$

$$S_{a;b}^{:b} = -(l^2 + l - 1) S_a$$

$$(S_{a;b} + S_{b;a})^{:c} = - [\quad] (S_{a;b} + S_{b;a})$$

$$(S_{a;b} + S_{b;a})^{:b} = [\quad] S_a$$

PROBLEM: FILL IN THE EMPTY SPACES, HINT: USE

$$V_{a;b;c} = V_{a;c;b} + V_d R^d_{abc}$$

where $R^d_{abc} = \delta^d_b \delta_{ac} - \delta^d_c \delta_{ab}$

completeness of harmonics on S^2

The harmonics listed on pages 25-26 constitute a set of orthonormal harmonics complete on S^2 . In other words any suitably behaved function, be a scalar, vector or tensor field, ~~can~~ on S^2 can be represented in terms of these harmonics

Scalar field :

Let $f(\theta, \varphi)$ be a scalar field on S^2 . Then

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} f_{lm} Y_{lm}(\theta, \varphi)$$

where f_{lm} are constants independent of θ and φ .

Vector field

Let $V_a(\theta, \varphi)$ be a vector field on S^2 . Then

$$V_a(\theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{+l} V_{lm}^{(e)} Y_{lm}(\theta, \varphi) + \sum_{l=1}^{\infty} \sum_{m=-l}^{+l} V_{lm}^{(o)} S_a^{lm}(\theta, \varphi)$$

where $V_{lm}^{(e)}$ are the even and $V_{lm}^{(o)}$ are the odd parity

coefficients,

(2)

Tensor field

Let $h_{ab}(\theta, \varphi)$ be a symmetric tensor field.

$$\text{Let } S_{ab}^{lm} \equiv S_{a|b}^{lm} + S_{b|a}^{lm}$$

$$Y_{ab}^{lm} \equiv Y_{a|b}^{lm} + l(l+1) Y_{ab}^{lm}$$

Then

$$h_{ab}(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{ab}^{lm} + \sum_{l=2}^{\infty} \sum_{m=-l}^l (b_{lm} Y_{ab}^{lm} + c_{lm} S_{ab}^{lm})$$

a_{lm} refers to the pure trace part of h_{ab}

b_{lm} " " trace-less even part of h_{ab}

c_{lm} " " (trace-less) ~~odd~~ part of h_{ab}

Scalar, Vector and Tensor fields on 30 Spherically Symmetric Space-Times.

We have now the technology of dealing with arbitrary scalar, vector, and tensor field on four-dimensional spherically symmetric space times, which we know has a metric of the form

$$ds^2 = g_{AB}(x^a) dx^A dx^B + r^2(x^a) \gamma_{ab} dx^a dx^b.$$

A.) Scalar field.

Consider ~~some~~ the scalar field

$$f(x^0, x^1, \theta, \varphi) = f(x^a, \theta, \varphi)$$

The completeness of the scalar harmonics on S^2 allows us to write

$$f(x^a, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_{lm}(x^a) Y_{lm}(\theta, \varphi)$$

where $f_{lm}(x^a)$ are expansion coefficients independent of θ and φ . These $f_{lm}(x^a)$, being functions of x^0 and x^1 , are therefore a set of scalars on the manifold M^2

B.) Vector field

Consider some ^{co}vector field

$$\{v_\mu(x^a, \theta, \phi)\} = \{v_A(x^a, \theta, \phi), v_a(x^a, \theta, \phi)\}$$

on space time a spherically symmetric

Here $A = (0, 1)$

and $a = (2, 3) = (\theta, \phi)$

(i) Observe that each vector component v_0 and v_1 is a scalar on S^2 . Consequently one can

Use scalar harmonics

$$v_A(x^a, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} v_A^{\ell m}(x^a) Y^{\ell m}(\theta, \phi)$$

(ii) Also observe that v_a is a vector field on S^2 .

Thus

$$v_a(x^a, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} v_{\ell m}^e(x^a) Y_{,a}^{\ell m}(\theta, \phi) + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} v_{\ell m}^o(x^a) S_a^{\ell m}$$

Here $v_{\ell m}^e(x^a)$ is the expansion coefficient for "even parity" and $v_{\ell m}^o(x^a)$ is the expansion coefficient for "odd" parity.

ii) Thus one had the following conclusion: ^{there is} Associated (32) with an arbitrarily given vector field v_μ a vector field $v_A^{\ell m}(x^q)$ and two scalar fields $v_{\ell m}^e(x^q)$ and $v_{\ell m}^o(x^q)$ on the two dimensional manifold M^2 spanned by x^q .

$$v_\mu \longleftrightarrow \begin{cases} v_A^{\ell m}(x^q) & \text{vector field } -\ell \leq m \leq \ell, \ell=0,1, \\ v_{\ell m}^e(x^q) & \text{scalar field } -\ell \leq m \leq \ell, \ell=1,2 \\ v_{\ell m}^o(x^q) & \text{scalar field } -\ell \leq m \leq \ell, \ell=1,2 \end{cases}$$

all on M^2

c.) Symmetric tensor field.

Consider an arbitrarily given symmetric 2nd rank tensor field $h_{\mu\nu}(x^\sigma) dx^\mu \otimes dx^\nu$ whose components are

$$h_{\mu\nu}(x^q, \theta, \varphi) = \begin{bmatrix} h_{AB} & h_{Ab} \\ h_{aB} & h_{ab} \end{bmatrix}$$

$$h_{\mu\nu} dx^\mu dx^\nu = h_{AB} dx^A dx^B + h_{Ab} (dx^A dx^b + dx^b dx^A) + h_{ab} dx^a dx^b$$

It is clear that

h_{AB} $A, B = \begin{cases} 0 \\ 1 \end{cases}$ are scalar fields on S^2

h_{Ab} $A = \begin{cases} 0 \\ 1 \end{cases}$ are vector fields on S^2

h_{ab} is a tensor field on S^2 ,
that are parametrized by

Thus one has in terms of scalar, vector and tensor harmonic x^0 and x^1

$$h_{AB}(x^0, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l h_{AB}^{lm}(x^0) Y^{lm}(\theta, \varphi)$$

$$h_{Ab} = \sum_{l=1}^{\infty} \sum_{m=-l}^l h_A^{lm}(x^0) Y_{,b}^{lm} + \sum_{l=1}^{\infty} \sum_{m=-l}^l h_A^{\sigma}(x^0) S_a^{lm}$$

$$h_{ab} = \sum_{l=0}^{\infty} \sum_{m=-l}^l r^2(x^0) K^{lm}(x^0) Y_{,ab}^{lm} +$$

$$+ \sum_{l=0}^{\infty} \sum_{m=-l}^l r^2 G^{lm}(x^0) Y_{,a;b}^{lm} +$$

$$+ \sum_{l=2}^{\infty} \sum_{m=-l}^l h(x^0) (S_{a;b} + S_{b;a})$$

COMMENT: To conform with the by-now-universally accepted convention introduced by Regge and Wheeler, Phys. Rev. 108 1063 (1957) we

introduce explicitly the factor r^2 into the expansion coefficients for Y_{ab} and $Y_{a;b}$. If $h_{\mu\nu} dx^\mu dx^\nu$ is a perturbation in the background metric then ~~the~~ introducing r^2 simplifies, as it turns out, the various harmonic components of the Einstein field equations.

Linearized

COMMENT 2: Instead of expanding $h_{ab}(x^\sigma)$ in terms of the orthogonal harmonics Y_{ab} and $Y_{ab} = Y_{a;b} + L(L+1)Y_{ab}$, as was suggested on P 29, it is permissible to expand $h_{ab}(x^\sigma)$ in and sometimes more convenient terms of the non-orthogonal but also linearly independent harmonics Y_{ab} and $Y_{a;b}$.

CONCLUSION: Given a symmetric tensor field $h_{\mu\nu} dx^\mu \otimes dx^\nu$, one can associate with it scalar, vector and tensor fields on M^2 as follows

	TYPE OF FIELD	PARITY	BASIS HARMONICS	
$h_{\mu\nu} \leftrightarrow$	$\frac{h_{AB}(x^a)}{r^2}$	tensor	Y_{lm}	
	$\frac{h^e_A(x^a)}{r^2}$	vector	$Y_{l,a}$	
	$\frac{r^2 K(x^a)}{r^2}$	1st scalar	$Y_{l,ab}$	
	$\frac{r^2 G(x^a)}{r^2}$	2nd scalar	$Y_{l,a;b}$	
	$\frac{h^o_A(x^a)}{r^2}$	vector	S_a	
	$\frac{h(x^a)}{r^2}$	scalar	$(S_{a;b} + S_{b;a})$	
			"EVEN"	
			"ODD"	

GEOMETRICAL OBJECTS ON M^2

NOTE: Reference to the angular integer l and m has been suppressed.

Example: (a) Write down the most general odd parity perturbation in the metric which is characterized by some integers l and m .

Answer:

$$h_{\mu\nu} dx^\mu dx^\nu = h^o_A S_a (dx^a \otimes dx^a + dx^a \otimes dx^a) + h(S_{a;b} + S_{b;a}) dx^a \otimes dx^b$$

(reference to l and m have been suppressed)

(b) Write down that most general even (36) parity perturbation in the metric which is characterized by l and m .

Answer:

$$h_{\mu\nu} dx^\mu \otimes dx^\nu = h_{AB} Y dx^A \otimes dx^B + h_{A,a}^e \left(dx^A \otimes dx^a + dx^a \otimes dx^A \right) \\ + \left(r^2 K Y \gamma_{ab} + r^2 G Y_{,a;b} \right) dx^a \otimes dx^b$$

(reference to l and m have been suppressed)

(c) Write down that most general odd parity perturbation in some vector field which is characterized by l and m .

Answer:

$$\xi_\mu(x^\sigma) dx^\mu = \xi^a(x^\sigma) S_a dx^a$$

(d) Same for even parity

Answer:

$$\xi_\mu(x^\sigma) dx^\mu = \xi_A^{\#}(x^A) Y_{(A,P)} dx^A + \xi^e(x^a) Y_{,a} dx^a$$

COMMENT: The coefficient of the various harmonics in examples a-d, i.e.,

a: $h_A dx^A$; h

b: $h_{AB} dx^A \otimes dx^B$; $h_A^e dx^A$; $r^2 K$; $r^2 G$

c: ξ^a

d: $\xi_A dx^A$; ξ^e

are all geometrical objects on M^2 whose metric is $ds^2 = g_{AB} dx^A \otimes dx^B$.

Thus in discussing scalar, vector or tensor fields on 4-dimensional space-time one may just as well use the geometrical objects (on M^2) for fixed integers l and m . There is no loss in generality in doing this because the scalar, vector or tensor fields on 4-D space-time can always be synthesized from its harmonic components.

Gauge Invariant Geometrical Objects

38

We are interested in considering in 4-D spacetime the following fields

$g_{\mu\nu}$	whose perturbations are	$h_{\mu\nu}$
$t_{\mu\nu}$		$\Delta t_{\mu\nu}$
ψ_μ		$\Delta \psi_\mu$
ϕ		$\Delta \phi$

Now these perturbations can in fact represent actual changes in the fields themselves, or they can represent merely gauge changes due to mere ~~or~~ infinitesimal coordinate changes associated with

$$x^\mu \rightarrow x^\mu + \xi^\mu$$

In the latter case some perturbation $h_{\mu\nu}, \Delta t_{\mu\nu}$ etc. would undergo a change given by

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu}) \\ \Delta t_{\mu\nu} &\rightarrow \Delta t_{\mu\nu} - (t_{\mu\nu;\sigma} \xi^\sigma + t_{\sigma\nu} \xi_{;\mu}^\sigma + t_{\mu\sigma} \xi_{;\nu}^\sigma) \\ \Delta \psi_\mu &\rightarrow \Delta \psi_\mu - (\psi_{\mu;\sigma} \xi^\sigma + \psi_\sigma \xi_{;\mu}^\sigma) \\ \Delta \phi &\rightarrow \Delta \phi - \phi_{;\sigma} \xi^\sigma \end{aligned}$$

(39)

Such changes are not merely annoying and cumbersome, but can, and in fact do, frustrate progress in perturbation relativistic theory.

The reason is clear: these gauge changes are arbitrary and unphysical because they do not reflect any actual geometrical changes but instead reflect merely a person's whimsical specification of a coordinate system. This was captured by ~~Eddington~~ Infeld who said that gravitational waves, i.e. perturbations in the metric tensor propagate not with the speed of light but rather with the speed of thought.

Our object is to somehow deal with perturbational quantities that remain invariant under gauge changes and which represent only geometrical changes in the fields themselves.

This goal one can accomplish by working with the geometrical objects (on M^2) associated with a given harmonic component of a perturbation.

Roughly speaking we shall do the following

- (i) exhibit the gauge changes on P 38 in terms of geometric objects on M^2 .
- (ii) perform linear operations (differentiation, addition, multiplication) on the ~~of~~ gauge changed perturbations and obtain after taking suitable linear combinations geometric objects (on M^2) that are gauge invariant under gauge transformation.

Remember that all of this will be done for a particular harmonic perturbation characterized by fixed angular integers l and m , which we shall suppress totally.

GAUGE CHANGES.

Consider the gauge changes induced by an infinitesimal coordinate change induced by the covector field (suppress l and m):

case 1: $\xi_\mu dx^\mu = \xi(x^a) S_a dx^a$ (odd parity)

case 2: $\xi_\mu dx^\mu = \xi_A Y dx^A + \xi_{,a} Y dx^a$ (even parity)

It is evident from P38 that we must find

$\xi_{\mu;\nu}$

Remembering or computing the Christoffel symbols

$$\Gamma_{\mu\nu}^\sigma: \Gamma_{AB}^C; \Gamma_{Aa}^b = v_A \delta_a^b; \Gamma_{ab}^A = -v^A g_{ab} \equiv -v^A r^2 \gamma_{ab}$$

(where $v_A \equiv r_{,A}/r$)

one has

case 1 (odd parity)

$$\xi_{\mu;\nu} \Rightarrow \begin{cases} \xi_{A;B} = 0 \\ \xi_{A;a} = -\xi v_A S_a \\ \xi_{a;A} = (\xi_{,A} - \xi v_A) S_a \\ \xi_{a;b} = \xi S_{a;b} \end{cases}$$

(odd parity cont'd)

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} : \left\{ \begin{array}{l} r^2 (\xi/r^2)_{,A} S_a \\ \xi (S_{a:b} + S_{b:a}) \end{array} \right.$$

case 2 (even parity):

$$\begin{aligned} \xi_{\mu;\nu} : \quad \xi_{A;B} &= \xi_{A|B} Y \\ \xi_{A;a} &= (\xi_A - \xi v_A) Y_{,a} \\ \xi_{a;A} &= (\xi_{,A} - \xi v_A) Y_{,a} \\ \xi_{a;b} &= \xi Y_{,a;b} + r^2 \xi_B v^B Y_{ab} \end{aligned}$$

$$\begin{aligned} \xi_{\mu;\nu} + \xi_{\nu;\mu} : \quad & \left. \begin{array}{l} (\xi_{A|B} + \xi_{B|A}) Y \\ [\xi_A + r^2 (\xi/r^2)_{,A}] Y_{,a} \\ 2 \xi Y_{,a;b} + 2 r^2 \xi_B v^B Y_{ab} \end{array} \right\} \text{(even parity)} \end{aligned}$$

The correspondingly changed metric perturbations

are

case 1 (odd parity):

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu}) \leftrightarrow \left. \begin{array}{l} \bar{h}_{AB} = 0 \\ \bar{h}_{Aa} = [h_A - r^2 (\xi/r^2)_{,A}] S_{,a} \\ \bar{h}_{ab} = [h - \xi] (S_{a:b} + S_{b:a}) \end{array} \right\} \text{ODD PARITY}$$

case 2 (even parity)

(4)

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu}) \leftrightarrow \bar{h}_{AB} = [h_{AB} - (\xi_{A|B} + \xi_{B|A})] Y$$

$$h_{Aa} = [h_A - \xi_A - r^2 (\xi/r^2)_{,A}] Y_{,a}$$

$$\bar{h}_{ab} = [r^2 K - 2 r^2 \xi_B v^B] Y Y_{ab}$$

$$+ [r^2 G - 2 \xi] Y_{,a;b}$$

Thus we have the following conclusion:
given an infinitesimal coordinate transformation of type L, m

$$\xi_\mu dx^\mu = \xi S_a dx^a \quad (\text{odd parity})$$

$$\xi_\mu dx^\mu = \xi_A Y dx^A + \xi Y_{,a} dx^a \quad (\text{even parity})$$

then the geometrical objects on M^2 which are associated respectively with an odd or an even parity metric perturbation are changed and assume the new values

$$\left. \begin{aligned} \bar{h}_A &= h_A - r^2 (\xi / r^2)_{,A} \\ \bar{h} &= h - \xi \end{aligned} \right\} \text{odd parity}$$

$$\left. \begin{aligned} \bar{h}_{AB} &= h_{AB} - (\xi_{A|B} + \xi_{B|A}) \\ \bar{h}_A &= h_A - \xi_A - r^2 (\xi / r^2)_{,A} \\ \bar{K} &= K - 2 \xi_B v^B \\ \bar{G} &= G - 2 \xi r^{-2} \end{aligned} \right\} \text{even parity}$$

NOTE: IF $P_A \equiv h_A - \frac{1}{2} r^2 G_{,A}$ then $\bar{P}_A = P_A - \xi_A$

The gauge transformed perturbation objects are frequently specified indirectly by imposing gauge conditions.

Example 1. (odd parity)
Find $\xi_\mu dx^\mu$ such that the metric perturbation has the form

$$\bar{h}_{\mu\nu} = \begin{bmatrix} 0 & h_A S_a \\ h_A S_a & 0 \ 0 \\ 0 \ 0 & 0 \ 0 \end{bmatrix}$$

if $h_{\mu\nu} = \begin{bmatrix} 0 & h_A S_a \\ h_A S_a & h (S_{a|b} + S_{b|a}) \end{bmatrix}$ is given,

Solution:

Let $\xi_\mu dx^\mu = \xi S_a dx^a$ be that "gauge vector field" which satisfies

$$0 = \bar{h} = h - \xi \Rightarrow \xi = h$$

Thus

$$\bar{h}_{\mu\nu} dx^\mu \otimes dx^\nu = \overbrace{[h_A - r^2(h/r^2)_{,A}] S_a}^{\bar{h}_A} (dx^a \otimes dx^a + dx^i \otimes dx^i)$$

Comment: If the metric perturbation has this form, it is said to be represented in the Regge-Wheeler gauge. Thus the R-W gauge for ^{the} odd-parity perturbation mode (characterized by l and m) is characterized by

$$\bar{h} = 0,$$

Example 2. (even parity)

Given:

$$h_{\mu\nu} = \begin{bmatrix} h_{AB} Y & h_A Y_{,a} \\ h_A Y_{,a} & -2K Y_{ab} + r^2 G Y_{,a;b} \end{bmatrix}$$

Find the metric perturbation represented in ⁽⁴⁶⁾ the R-W gauge, i.e. so that

$$\bar{h}_{\mu\nu} = \left[\begin{array}{c|c} \bar{h}_{AB} \gamma & 0 \\ \hline 0 & r^2 \bar{K} \gamma_{ab} \end{array} \right]$$

i.e. so that $\bar{h}_A = 0$
 $\bar{G} = 0$

Solution:

$$\bar{G} = 0 \Rightarrow \frac{1}{2} G = \xi / r^2$$

$$\bar{h}_A = 0 \Rightarrow \xi_A = h_A - r^2 (\xi / r^2)_{,A} = h_A - \frac{1}{2} r^2 G_{,A}$$

from above

Thus

$$\bar{h}_{\mu\nu} dx^\mu \otimes dx^\nu = \left[h_{AB} - (P_{A|B} + P_{B|A}) \right] dx^A \otimes dx^B + r^2 (K - 2 P_B v^B) \gamma_{ab} dx^a \otimes dx^b$$

where we set $P_A = h_A - \frac{1}{2} r^2 G_{,A}$

Comment: The R-W gauge for even parity $l=m$ characterized metric perturbations is characterized by

$$\boxed{\bar{h}_A = 0} \quad \text{and} \quad \boxed{\bar{G} = 0}$$

comment: There ^{obviously} exist other gauges besides the R-W gauge. For example, Lifschitz and Khalatnikov (Adv. Phys. 12, 185 (1963)) in considering perturbations on homogeneous isotropic space times with space like hypersurfaces, use ^{what} they call the synchronous gauge

$$\bar{h}_{00} = \bar{h}_{01} = \bar{h}_{02} = \bar{h}_{03} = 0$$

~~There is~~ Another gauge is the so-called Lorentz gauge,

$$\bar{h}_{\mu\nu};\nu = 0$$

which determined $\xi_{\mu} dx^{\mu}$ not by an algebraic relation, as is the case with the R-W gauge, but instead by a set of first order eq'ns for differentia $\xi_{\mu} dx^{\mu}$.

Construction of gauge Invariant geometrical objects on M^2

(48)

Now we do step (ii) as mentioned on page 40,

Case 1: (odd parity)

For odd parity we have the following geometrical objects on M^2

$$\bar{h}_A = h_A - r^2 (\xi / r^2)_{,A} \quad (1)$$

$$\bar{h} = h - \xi \quad (2)$$

Divide (2) by r^2 , take $\frac{\partial}{\partial x^A}$, multiply by r^2 ,

then subtract from (1) to eliminate ξ term

The result is evidently

$$\bar{h}_A - r^2 (\bar{h} / r^2)_{,A} = h_A - r^2 (h / r^2)_{,A} \equiv k_A$$

which is a gauge invariant vector on M^2 .

It is characterized by the angular integers l and m , which have been suppressed.

Thus given an odd parity metric perturbation

one can always obtain a corresponding gauge invariant vector on M^2 .

Conversely, given a covector k_A on M^2 (49) which is to be associated with the integers l and m , one can recover, if one cares to, the original ~~is~~ corresponding metric perturbation provided one specifies the gauge with respect to which the perturbation is to be given; in other words k_A determines does not h_A and h (and hence $h_{\mu\nu}$) uniquely. Instead k_A determines $h_{\mu\nu}$ "modulo" an odd parity gauge transformation.

case 2: (even parity)

For even parity metric perturbation one has the following gauge transformed ~~met~~ geometrical objects on M^2 :

$$(1) \quad \bar{h}_{AB} = h_{AB} - (\xi_{A|B} + \xi_{B|A})$$

$$(2) \quad \bar{h}_A = h_A - \xi_A - r^2 (\xi / r^2)_{,A}$$

$$(3) \quad \bar{K} = K - 2\xi_B v^B$$

$$(4) \quad \bar{G} = G - 2\xi / r^2$$

$$(5) \quad \bar{P}_A = P_A - \xi_A \quad \text{where} \quad P_A \equiv h_A - \frac{1}{2} r^2 G_{,A}$$

NOTE: a) by introducing P_A we have, in effect, (5) already eliminated ξ between (2) & (4).
 b) We therefore use (5) to eliminate $\xi_{A|B}^+$ and ξ_B in (1) and (3) respectively. Result

$$\bar{h}_{AB} - (\bar{P}_{A|B} + \bar{P}_{B|A}) = h_{AB} - (P_{A|B} + P_{B|A}) \equiv k_{AB}$$

$$\bar{K} - 2\nu_B \bar{P}^B = K - 2\nu_B P^B = k$$

Thus, even parity metric perturbations are characterized by two gauge invariant geometrical objects, a symmetric tensor

on M^2

k_{AB} and the scalar k .

Conclusion: Given an even parity (l-m characterized) metric perturbation one can always obtain two corresponding gauge invariant objects on M^2 a symmetric tensor together with a scalar.

Gauge Invariant Geometrical Objects (II) (5)

So far we have constructed gauge invariant perturbation objects associated with the metric tensor $g_{\mu\nu}$. The results are given on P 48 and P 50 for odd and even parity respectively. Reference to P 38 shows that we must fulfill the analogous task

For also

on a spherically symmetric background	{	$t_{\mu\nu}$	arbitrary symmetric tensor field
		v_μ	vector field
		ϕ	scalar field

~~A) Arbitrary~~ unperturbed

A) In a spherically symmetric background these $t_{\mu\nu}$, v_μ and ϕ may not be arbitrary otherwise they would destroy the spherical symmetry. For example, these fields may be respectively the unperturbed stress energy tensor, unperturbed fluid

four velocity and the unperturbed pressure or (52) density. These quantities must reflect the spherical symmetry of the background which they are part of. A little reflection will convince one that none of them may depend on the angular coordinates θ and ϕ . Not only that, but certain tensor and vector components must also be absent. In

fact

$$t_{\mu\nu} dx^\mu dx^\nu = t_{AB}(x^A) dx^A dx^B + \frac{1}{2} t_a^a(x^A) \underbrace{g_{bc} dx^b dx^c}_{r^2 (d\theta^2 + \sin^2\theta d\phi^2)}$$

$$\text{or } t_{\mu\nu} = \begin{bmatrix} t_{AB}(x^A) & | & 0 & 0 \\ 0 & 0 & | & 0 & 0 \\ 0 & 0 & | & \frac{1}{2} t_a^a & g_{bc} \\ \vdots & \vdots & | & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} t_{00} & t_{01} & | & 0 & 0 \\ t_{10} & t_{11} & | & 0 & 0 \\ 0 & 0 & | & \frac{1}{2} t_a^a & 0 \\ 0 & 0 & | & 0 & \frac{1}{2} t_a^a \sin^2\theta \end{bmatrix}$$

where $\frac{1}{2} t_a^a = \frac{1}{2} (t_2^2 + t_3^3) = t_2^2 = t_3^3$ is the partial trace of $t_{\mu\nu}$.

Similarity for the vector field

$$w_m dx^m = w_A(x^a) dx^A$$

$$\text{or } w_m : \begin{bmatrix} w_0(x^a) \\ w_1(x^a) \\ 0 \\ 0 \end{bmatrix}$$

and for the scalar field

$$\phi(x^m) = \phi(x^a), \text{ i.e. as mentioned before,}$$

ϕ may only depend on the coordinates x^a ($a=0, 1$) that span the manifold M^2 .

Summary: The unperturbed geometrical objects $g_{\mu\nu}, t_{\mu\nu}, v_\mu$, and ϕ have associated with them the following geometrical objects on M^2

metric:

$$g_{\mu\nu} dx^\mu \otimes dx^\nu \leftrightarrow \left\{ \begin{array}{l} g_{AB}(x^a) dx^A \otimes dx^B \text{ (metric)} \\ r^2(x^a) \text{ (scalar)} \end{array} \right\} \text{ on } M.$$

symmetric tensor:

$$t_{\mu\nu} dx^\mu \otimes dx^\nu \leftrightarrow \left\{ \begin{array}{l} t_{AB}(x^a) dx^A \otimes dx^B \text{ (symm. tensor)} \\ \frac{1}{2} t_a^a(x^a) \text{ (scalar)} \end{array} \right\} \text{ on } M$$

vector:

$$W_\mu dx^\mu \leftrightarrow W_A(x^a) dx^A \text{ (vector) } \} \text{ on } M^2$$

(54)

scalar:

$$\varphi(x^\mu) \leftrightarrow \varphi(x^a) \text{ (scalar) } \} \text{ on } M^2.$$

B) The perturbations of the back ground fields $t_{\mu\nu}$, W_μ , and φ are expressible in terms tensor, vector and scalar harmonics. For a harmonic component given by specified by the integers l and m (which we suppress) one has for

(i) odd parity perturbations

$$\begin{aligned} \text{ii. tensor:} \\ \Delta t_{\mu\nu} dx^\mu dx^\nu = \Delta t_A(x^a) S_a (dx^A dx^a + dx^a dx^A) \\ + \Delta t (S_{a:b} + S_{b:a}) dx^a dx^b \end{aligned}$$

$$\Delta t_{\mu\nu}: \left[\begin{array}{cc|cc} 0 & 0 & \Delta t_0 S_a & \\ 0 & 0 & \Delta t_1 S_a & \\ \hline \text{Symm.} & & \Delta t (S_{a:b} + S_{b:a}) & \end{array} \right]$$

iii. vector:

$$\Delta W_\mu dx^\mu = \Delta W S_a dx^a$$

iiii. scalar: $\Delta \varphi = 0$! (for odd parity)

(2) even parity perturbations

i. tensor:

$$\begin{aligned} \Delta t_{\mu\nu} dx^\mu dx^\nu &= \Delta t_{AB} Y dx^A dx^B \\ &+ \Delta t_A Y_{,a} (dx^A dx^a + dx^a dx^A) \\ &+ \left[\Delta t^1 r^2 Y_{ab} + \Delta t^2 Y_{,a;b} \right] dx^a dx^b \end{aligned}$$

ii. vector:

$$\Delta w_\mu dx^\mu = \Delta w_A Y dx^A + \Delta w Y_{,a} dx^a$$

iii. scalar:

$$\Delta \bar{\phi} = \Delta \phi(x^a) Y$$

c) The gauge changes of these all of perturbations are given on P 38. These changes are linear in $\xi_\mu dx^\mu$ and in $\xi_{\mu;\nu}$, which are exhibited on P 41 and 42. With the help of these expressions the gauge changes are

(1) Odd parity

Tensor:

$$\Delta t_{\mu\nu} = \Delta t_{\mu\nu} - (t_{\mu\nu;\sigma} \xi^\sigma + t_{\sigma\nu} \xi^\sigma_{;\mu} + t_{\mu\sigma} \xi^\sigma_{;\nu})$$

whose odd parity coefficients are

$$\left. \begin{aligned} \overline{\Delta t_A} &= \Delta t_A - \frac{1}{2} t_a^a r^2 (\xi / r^2)_{,A} \\ \overline{\Delta t} &= \Delta t - \frac{1}{2} t_a^a \xi \end{aligned} \right\} \begin{array}{l} \text{(general} \\ \text{symmetric} \\ \text{tensor)} \end{array}$$

ii, vector:

$$\overline{\Delta w_\mu} = \Delta w_\mu - (w_{\mu;\sigma} \xi^\sigma + w_\sigma \xi_{;\mu}^\sigma)$$

~~general vector~~

whose odd parity coefficients are

$$\overline{\Delta w} = \Delta w - 0 \quad \left. \vphantom{\overline{\Delta w}} \right\} \text{i.e. } \Delta w \text{ is an odd parity gauge invariant}$$

iii, scalar!

$$\begin{aligned} \overline{\Delta \phi} &= \Delta \phi - \phi_{,\sigma} \xi^\sigma \\ \overline{\Delta \phi} &= 0 - 0 \end{aligned}$$

(2) Even parity

i, tensor!

$$\overline{\Delta t_{\mu\nu}} = \Delta t_{\mu\nu} - (t_{\mu\nu;\sigma} \xi^\sigma + t_{\sigma\nu} \xi_{;\mu}^\sigma + t_{\mu\sigma} \xi_{;\nu}^\sigma)$$

whose even parity coefficients are

$$\left. \begin{aligned} \overline{\Delta t_{AB}} &= \Delta t_{AB} - t_{ABIC} \xi^C - t_{CB} \xi_{IA}^C - t_{AC} \xi_{IB}^C \\ \overline{\Delta t_A} &= \Delta t_A - t_{AC} \xi^C - \frac{1}{2} r^2 t_a^a (\xi / r^2)_{,A} \\ \overline{\Delta t^1} &= \Delta t^1 - \frac{1}{2} r^{-2} (r^2 t_a^a)_{,A} \xi^A \\ \overline{\Delta t^2} &= \Delta t^2 - t_a^a \xi \end{aligned} \right\} \begin{array}{l} \text{(general} \\ \text{symmetric} \\ \text{tensor)} \end{array}$$

ii. vector:

(57)

$$\overline{\Delta w_\mu} = \Delta w_\mu - (w_{\mu;\sigma} \xi^\sigma + w_\sigma \xi^\sigma{}_{;\mu})$$

whose even parity coefficients are:

$$\left. \begin{aligned} \overline{\Delta w_A} &= \Delta w_A - w_{A|B} \xi^B - w^B \xi_{B|A} \\ \overline{\Delta w} &= \Delta w - w_A \xi^A \end{aligned} \right\} \text{(general vector)}$$

iii. scalar:

$$\overline{\Delta \phi} = \Delta \phi - \phi_{;\sigma} \xi^\sigma \quad \text{(scalar)}$$

whose even parity coefficients are

$$\overline{\Delta \phi} = \Delta \phi - \phi_{;B} \xi^B$$

D) Gauge invariants associated with (1) i.-iii. and with (2) i.-iii. are obtained by eliminating ξ , ξ_A , ξ^{odd} , ξ^{even} etc between these expressions and those associated with metric perturbation gauge ~~ga~~ changes (P 44):

$$\left. \begin{aligned} \overline{h}_A &= h_A - r^2 (\xi / r^2)_{;A} \\ \overline{h} &= h - \xi \end{aligned} \right\} \text{odd parity}$$

$$\bar{h}_A - \frac{1}{2} r^2 \bar{G}_{,A} \equiv \bar{P}_A = P_A - \xi_A \quad (\equiv h_A - \frac{1}{2} r^2 G_{,A} - \xi_A)$$

$$\frac{1}{2} r^2 \bar{G} = \frac{1}{2} r^2 G - \xi$$

} even parity

This is achieved by taking suitable linear combinations. The result evidently is

(1) Odd parity:

i. tensor:

$$t_{\mu\nu} \rightarrow \begin{cases} \Delta \bar{t}_A - \frac{1}{2} t_a^a \bar{h}_A = \Delta t_A - \frac{1}{2} t_a^a h_A \equiv T_A \\ \Delta \bar{t} - \frac{1}{2} t_a^a \bar{h} = \Delta t - \frac{1}{2} t_a^a h \equiv T_A \end{cases}$$

ii. vector:

$$w_\mu \rightarrow \{ \Delta \bar{w} = \Delta w = W \}$$

iii. scalar:

$$\phi \rightarrow \text{zero}$$

(2) Even parity:

i. tensor:

$$t_{\mu\nu} \rightarrow \begin{cases} \Delta \bar{t}_{AB} - \bar{t}_{ABIC} \bar{P}^d - t_A^c \bar{P}_{CIB} - t_B^c \bar{P}_{CIA} \\ = \Delta t_{AB} - t_{ABIC} \bar{P}^c - t_A^c \bar{P}_{CIB} - t_B^c \bar{P}_{CIA} \equiv T_A \end{cases}$$

cont'd on next page

$$\begin{aligned}
 t_{\mu\nu} \rightarrow & \left\{ \begin{aligned}
 \overline{\Delta t_A} - \overline{t_A} \overline{p^C} - \frac{1}{4} r^2 \overline{t_a^a} \overline{G_{,A}} &= \\
 &= \Delta t_A - t_{AC} p^C - \frac{1}{4} r^2 t_a^a G_{,A} \equiv T_A \\
 \overline{\Delta t^1} - \frac{1}{2} r^2 (\overline{r^2 t_a^a})_{,B} \overline{p^B} &= \\
 &= \Delta t^1 - \frac{1}{2} r^2 (r^2 t_a^a)_{,B} p^B \equiv T^1 \\
 \overline{\Delta t^2} - \frac{1}{2} r^2 \overline{t_a^a} \overline{G} &= \Delta t^2 - \frac{1}{2} r^2 t_a^a G \equiv T^2
 \end{aligned} \right.
 \end{aligned}$$

ii, vector:

$$\begin{aligned}
 w_\mu \rightarrow & \left\{ \begin{aligned}
 \overline{\Delta w_A} - \eta_{A|B} \overline{p^B} - \eta_B \overline{p^B}_{,A} &= \Delta w_A - \eta_{A|B} p^B - \eta_B p^B_{,A} \equiv | \\
 \overline{\Delta w} - w_{B,} \overline{p^B} &= \Delta w - w_B p^B \equiv W
 \end{aligned} \right.
 \end{aligned}$$

iii, scalar:

$$\Phi \rightarrow \left\{ \overline{\Delta \Phi} - \Phi_{,B} \overline{p^B} = \Delta \Phi - \Phi_{,B} p^B \equiv \Phi \right.$$

NOTICE: if $t_{\mu\nu} = g_{\mu\nu}$ then the odd and even gauge invariant reduce to the metric perturbation gauge invariants listed on P 48 and 50 respectively.

also NOTICE that all gauge invariants constructed
(are geometrical objects (tensors, vectors,
scalars) on the two dimensional space-time
 M^2 .

The Linearized Einstein Field Equations.

We must now take the linearized Einstein field equations (P10), consider a particular harmonic component of the perturbation for the metric

$$h_{\mu\nu} \rightarrow \left[\begin{array}{c|c} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & h_A S_a \\ \hline h_A S_a & h (S_{a;b} + S_{b;a}) \end{array} \right] \text{ for odd parity}$$

and

$$h_{\mu\nu} \rightarrow \left[\begin{array}{c|c} h_{AB} Y & h_A Y_{,a} \\ \hline h_A Y_{,a} & r^2 K Y \gamma_{ab} + r^2 G Y_{,a;b} \end{array} \right] \text{ for even parity}$$

and of the stress energy tensor

$$\Delta t_{\mu\nu} \rightarrow \left[\begin{array}{c|c} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \Delta t_A S_a \\ \hline \Delta t_A S_a & \Delta t (S_{a;b} + S_{b;a}) \end{array} \right] \text{ for odd parity}$$

and

$$\Delta t_{,\mu\nu} \rightarrow \left[\begin{array}{c|c} \Delta t_{AB} Y & \Delta t_A Y_{,a} \\ \hline \Delta t_A Y_{,a} & \Delta t^1 r^2 Y \gamma_{ab} + \Delta t^2 Y_{,a;b} \end{array} \right] \text{ for even parity}$$

substitute these perturbations into the linearized field equations (P10) and use the expressions for R , R_{AB} , $R_{\alpha\beta}$ and $R_{\alpha\beta}$ on P7 ~~in the~~ in the section on generic spherically symmetric space times. It should be clear ~~that~~ (and this follows from the properties of the spherical scalar, vector and tensor harmonics) that the linearized field equations separate into equations that involve expressions proportional to l .

$$\text{and } \left. \begin{array}{l} S_a \quad l=1, 2, \dots \\ (S_{a;b} + S_{b;a}) \quad l=2, 3, \dots \end{array} \right\} \text{for odd parity}$$

$$\left. \begin{array}{l} Y \quad l=0, 1, 2, \dots \\ Y_{,a} \quad l=1, 2, \dots \\ Y_{,a;b} \quad l=1, 2, \dots \\ Y_{\alpha\beta} \quad l=0, 1, 2, \dots \end{array} \right\} \text{for odd parity}$$

(6)
These harmonics are linearly independent.
Consequently the coefficients of these harmonics on one side of the equation must equal the coefficients of the same harmonics on the other side of the equation.

The results ing coefficients are obtained by a systematic computation. As usual, we suppress explicit reference to the angular integers l and m .
A table of the relevant first derivatives is useful.

Here we list the derivatives $h_{\mu\nu;\sigma}$ of a given perturbation mode l, m for odd parity, Eq. (5.1). The reduced form of the non-zero Christoffel symbols that is the basis of our perturbational formalism is

$$\Gamma_{\mu\nu}^{\sigma} : \Gamma_{AB}^C ; \Gamma_{Aa}^b = v_A \delta_a^b ; \Gamma_{ab}^A = -v^A g_{ab} = -v^A r^2 \gamma_{ab}.$$

Here Γ_{AB}^C and $v_A = \gamma_{,A}/r$ are defined on the totally geodesic submanifold M^2 whose metric is

$$g_{AB} dx^A dx^B \quad (A, B = 0, 1)$$

The derivative $h_{\mu\nu;\sigma}$ themselves have the form

Odd parity:

$$h_{AB;c} = 0$$

$$h_{Aa;b} = (h_A{}_{|B} - h_A v_B) S_a$$

$$h_{AB;a} = -(h_A v_B + h_B v_A) S_a$$

$$h_{Aa;b} = h_A S_{a:b} - v_A h (S_{a:b} + S_{b:a})$$

$$h_{ab;A} = (h_{,A} - 2v_A h) (S_{a:b} + S_{b:a})$$

$$h_{ab;c} = v^c h_c (S_a g_{bc} + S_b g_{ac}) + h (S_{a:b:c} + S_{b:a:c})$$

Here we list the derivatives $h_{\mu\nu;\sigma}$ of a given perturbation mode ℓ, m for even parity, Eq. ~~42~~ ^{on p 61}. The reduced form of the non-zero Christoffel symbols that is the basis of our perturbational formalism is

$$\Gamma_{\mu\nu}^{\sigma} : \Gamma_{AB}^C ; \Gamma_{Aa}^b = v_A \delta_a^b ; \Gamma_{ab}^A = -v^A g_{ab} \equiv v^A r^2 \gamma_{ab}$$

Here Γ_{AB}^C and $v_A = r_{,A}/r$ are defined on the totally geodesic submanifold M^2 whose metric is

$$g_{AB} dx^A dx^B \quad A, B = 0, 1$$

The derivatives $h_{\mu\nu;\sigma}$ themselves have the form

$$h_{AB;c} = h_{ABc} Y + 0 + 0 + 0$$

$$h_{aB;c} = 0 - (h_{Bc} - h_B v_c) Y_{,a} + 0 + 0$$

$$h_{AB;a} = h_{AB} Y_{,a} - (h_A v_B + h_B v_A) Y_{,a} + 0 + 0$$

$$h_{ab;B} = 0 + 0 + K_{,B} \gamma_{ab} + r^2 G_{,B} Y_{,a:b}$$

$$h_{aB;b} = v^A h_{AB} \gamma_{ab} + h_B Y_{,a:b} - K v_B \gamma_{ab} - r^2 v_B G Y_{,a;b}$$

$$h_{ab;c} = 0 + v^c h_c (Y_{,a} \gamma_{bc} + Y_b \gamma_{ac}) + K \gamma_{ab} Y_{,c} + r^2 G Y_{,a:b}$$

(6)

Odd Parity Linearized field equations on spherical space time

Equating the above mentioned coefficients one has

$$S_{a;b} + S_{b;a} : k^c{}_{|c} = 8\pi T \quad (1) \quad \ell = 2, 3, \dots$$

$$S_a :$$

$$-\left[r^4 (r^{-2} k_A)_{|c} - r^4 (r^{-2} k_c)_{|A} \right]^{|c} + (\ell-1)(\ell+2) k_A = 8\pi r^2 T_A \quad (2) \quad \ell = 1, 2, 3, \dots$$

where

$$k_A \equiv h_A - r^2 (h/r^2)_{,A} \quad (\text{metric})$$

$$T_A \equiv \Delta t_A - \frac{1}{2} t_a{}^a h_A \quad \left. \vphantom{T_A} \right\} (\text{matter})$$

$$T \equiv \Delta t - \frac{1}{2} t_a{}^a h$$

are the gauge invariant perturbations object odd parity

on M^2 . Equations (1) and (2) are a scalar and a vector equation on M^2 . Their solution can be given in terms of the solution of a single master scalar wave type equation. This is done as follows.

Any antisymmetric tensor, ^(say F_{AB}) in two dimensions can be expressed in terms of a scalar π and the antisymmetric (Levi Civita) unit tensor

$$F_{AB} = \pi \epsilon_{AB} \quad (F_{AB} = -F_{BA}) \quad (3)$$

where

$$\pi = \frac{1}{2!} F_{CD} \epsilon^{CD}$$

Now let

$$F_{AB} = (r^{-2} k_A)_{,B} - (r^{-2} k_B)_{,A};$$

take the curl of the vector eq'n (2), ~~and~~

Note: the curl of a vector W_A is $W_{[AB]} = \frac{1}{2} [W_{A|B} - W_{B|A}]$, which is an antisymmetric tensor.

use Eq. (3) and obtain

$$2 \left\{ r^{-2} (r^4 \pi \epsilon_{C[A})^{,C} \right\}_{,B]} + (\ell-1)(\ell+2) \pi \epsilon_{AB} = 8\pi S \epsilon_{AB}$$

where

$$S = \frac{1}{2!} T_{AC} \epsilon^{AC}$$

Now use $\epsilon_{AIB} = 0$ and decompose the 1st term in a way analogous to Eq. (3), and obtain

$$\frac{2}{2!} \left\{ r^{-2} (\tau^4 \pi)^{IC} \right\}_{IF} \underbrace{\epsilon_{CE} \epsilon^{EF}}_{-\delta_c^F} \epsilon_{AB} + (\ell-1)(\ell+2) \pi \epsilon_{AB} = 8\pi S \epsilon_{AB}$$

which reduces to the master eq'n

$-\left\{ r^{-2} (\tau^4 \pi)^{IC} \right\}_{IC} + (\ell-1)(\ell+2) \pi = 8\pi S$	SCALAR MASTER EQ'N
---	--------------------------

here
$$\pi = \frac{1}{2} \left[(\tau^{-2} k_{,A})_{IB} - (\tau^{-2} k_B)_{,IA} \right] \epsilon^{AB}$$

The gauge invariant vector k_A can be obtained from Eq. (7), provided $\ell \geq 2$

$$k_A = \left[8\pi r^2 T_A + \epsilon_{AC} (\tau^4 \pi)^{IC} \right] \left[(\ell-1)(\ell+2) \right]^{-1} \quad \ell \geq 2$$

from which one can, if one cares to, obtain the odd parity metric perturbations.

By the well known property

$$W^A{}_{IA} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^A} (g^{AB} \sqrt{-g} W_B) \quad \left(\text{Problem: prove this identity} \right)$$

The scalar master equation defined (67)
on M^2 , two dimensional space-time, with
metric

$$g_{AB} dx^A dx^B,$$

becomes

$$-\frac{1}{\sqrt{-g}} \left\{ \frac{\partial}{\partial x^A} \left[g^{AB} \sqrt{-g} r^{-2} (\gamma^A \pi)_{,B} \right] + (\ell-1)(\ell+2)\pi \right\} = 8\pi S$$

$$\ell = 2, 3, 4, \dots$$

Some unsolved mathematical and physical problems.

(68)

Problem 1:

Stress energy tensor for gravitational waves

In the limit of wavelengths small compared to the background radius of curvature or in the limit that the background is flat one can consider the effective stress-energy tensor associated with gravitational waves. This effective stress energy tensor is given by

$$T_{\mu\nu}^{(GW)} = \frac{1}{32\pi} \left\langle \bar{h}_{\alpha\beta;\mu} \bar{h}^{\alpha\beta}{}_{;\nu} - \frac{1}{2} \bar{h}_{\alpha\beta;\mu} \bar{h}^{\alpha\beta}{}_{;\nu} - 2 \bar{h}^{\alpha\beta}{}_{;\mu} \bar{h}_{\alpha\beta;\nu} \right\rangle$$

where $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h^\gamma{}_\gamma$

where ";" denotes derivative w.r.t. the background metric,

and $\langle \dots \rangle$ denotes an average over several wavelengths (see MTW chapter 35)

Problem 1 (cont'd)

a) In terms of the odd parity gauge invariant

k_A find T_{AB}, T_{Aa}, T_{ab} associated with odd parity gravitational radiation.

b) In terms of the even parity gauge invariant

k_{AB}, k find T_{AB}, T_{Aa}, T_{ab} associated with even parity gravitational radiation.

(Suggestion: is the table of first derivatives on P63a and 63b of any help?)

Problem 2: Non-spherical oscillations (70)
of a relativistic star composed
of a uniform density fluid.

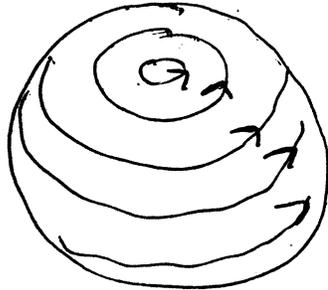
Consider a liquid drop of stellar size
composed of incompressible uniform density
matter (See e.g. MTW ch 23) This is a rela-
tivistic model of a star in equilibrium.
It is unrealistic because pressure perturba-
tions would yield infinite sound velocity
($v = (dp/d\rho)^{1/2} \rightarrow \infty$). It does however have
the virtue of capturing the essentials of
general relativity as far as the existence
and non-existence of equilibrium con-
figurations is concerned.

It is possible to discuss, formulate, and
analyze perturbations of such a star,
provided one focusses attention on pertur-
bations that do not involve pressure
changes. Thus one can focus attention on

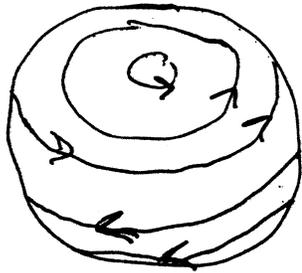
at least two types of perturbations

(71)

1. differential rotation:
(also uniform)



$l=1$

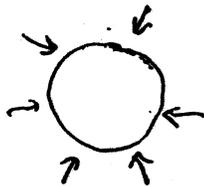


$l=2$

etc.

odd parity

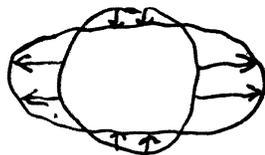
2. Convective motion:



$l=0$ (addition of mass)



$l=1$ (translation)



$l=2$ (quadrupole oscillation)

etc

even parity

For odd parity (as well as for even parity) find the perturbed stress energy tensor $t_{\mu\nu} \rightarrow t_{\mu\nu} + \Delta t_{\mu\nu}$

and from these the corresponding gauge invariant geometrical objects that constitute the matter source for the odd and the even parity gauge invariant linearized Einstein field equations. Note that the fluid may be a perfect fluid (no viscosity) or a fluid with viscosity in which case neutrino~~on~~ radiation is produced.

b) Set up the odd parity master wave equation for $l = 1, 2, 3, \dots$ as well as for $l = 2, 3, \dots$ and ~~find~~ find the metric gauge invariant at large distances.

When viscosity is present, compare the rate at which energy is carried away in the form of neutrinos with the rate at which energy is carried away in the form of gravitational radiation.

Problem 3: Torsional Oscillations of a
 Neutron Star composed of crystalline
 matter.

Consider a spherically symmetric star
 whose shear modulus is non-zero. Find
 the perturbations $\Delta t_{\mu\nu}$ away from spherical
 symmetry which describe odd parity
 shear (torsional) motion. Then find the
 odd parity ~~star~~ gauge invariant geometric
^{responding} perturbation object.

Write down the ^{two} coupled equations that
 describe the matter and the gravitational
 oscillatory degrees of freedom. This
 describes a neutron star suffering from a
 star quake.

Problem 4: Star quakes of a neutron star (74)
subsequent to collapse: parametrically excited torsional oscillations.

Consider the same star as in Problem 3, but instead of ~~the~~ ^{its} background metric being time independent, let its background metric vary periodically with time. Pick some particular time dependence. Now consider the coupled matter and ^(torsional) gravitational oscillatory degrees of freedom.

a) Assume G , the gravitational constant, is zero. This decouples the torsional from the gravitational wave degrees of freedom.

For frequencies of the specially symmetric background pulsations will the torsional oscillations grow? decay?

Problem 4: (cont'd)

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b) Now let $G \neq 0$. Will the gravitational degrees of freedom sap ~~out~~ energy out of the torsional degrees of freedom so that the system will always decay.

Problem 5: Even parity linearized Einstein field equations reduced to a master scalar equation. 76

Consider an l, m -characterized perturbation in the metric

$$h_{\mu\nu} dx^\mu dx^\nu = h_{AB}(x^a) Y dx^A dx^B + h_a Y_{,a} (dx^A dx^a + dx^a dx^A) + [\tau^2 K(x^a) Y \gamma_{ab} + \tau^2 G Y_{,a;b}] dx^a dx^b$$

and in the stress energy tensor

$$\Delta t_{\mu\nu} dx^\mu dx^\nu = \Delta t_{AB} Y dx^A dx^B + \Delta t_A Y_{,a} (dx^A dx^a + dx^a dx^A) + [\tau^2 \Delta t^1 Y \gamma_{ab} + \Delta t^2 Y_{,a;b}] dx^a dx^b$$

Construct the corresponding gauge invariant object on M^2

$$K_{AB} = h_{AB} - P_{AIB} - P_{BIA}$$

$$K = K - 2v^A p_A$$

} metric

where $v_A = \tau_{,A} / \tau$

$$p_A = h_A - \frac{1}{2} \tau^2 G_{,A}$$

Problem 5 (cont'd)

$$\left. \begin{aligned} T_{AB} &= \Delta t_{AB} - t_{AB}{}^{1c} p_c - t_A{}^c p_{cB} - t_B{}^c p_{cA} \\ T_A &= \Delta t_A - t_A{}^c p_c - r^2 (t_a{}^a/4) G_{,A} \\ T^1 &= \Delta t^1 - (p^c/r^2) (r^2 t_a{}^a/2)_{,c} \\ T^2 &= \Delta t^2 - (r^2 t_a{}^a/2) G \end{aligned} \right\} \text{matter}$$

Then the EVEN PARITY gauge invariant field equations are

$$-16\pi T^2 = k^c{}_c$$

$$-16\pi T_A = k_{,A} - k_{AB}{}^{1B} + k_C{}^c{}_{1A} - v_A k_C{}^c$$

$$\begin{aligned} -16\pi T^1 &= - \left\{ k_{,c}{}^{1c} + 2v^c k_{,c} + G_a{}^a k \right\} \\ &\quad + \left\{ k_{CD}{}^{1CID} + 2v^c k_{CD}{}^{1D} + 2(v^{c1D} + v^c v^D) k_{CD} \right\} \\ &\quad - \left\{ k_C{}^c{}_{1D} + v^c k_D{}^D{}_{1C} + R k_C{}^c - \frac{\ell(\ell+1)}{r^2} k_C{}^c \right\} \end{aligned}$$

$$-16\pi T_{AB} = r^{-2} \left[r^2 (k_{AB1C} - k_{AC1B} - k_{BC1A}) \right]^{1c} - \left[\frac{\ell(\ell+1)}{r^2} + G_C{}^C + G_a{}^a \right] k_{AB}$$

$$+ g_{AB} \left[r^{-2} (r^2 k_{CD})^{1CID} - G^{CD} k_{CD} \right] + k^c{}_{C1A1B}$$

$$- g_{AB} \left[r^{-2} (r^2 k_{G1D})^{1D} - \frac{\ell(\ell+1)}{r^2} k_G{}^G - \frac{1}{2} (G_C{}^C + G_a{}^a) k_G{}^G \right]$$

$$+ 2 [v_A k_{,B} + v_B k_{,A} + k_{,A1B}]$$

$$- g_{AB} \left[2k_C{}^c + 6v^c k_{,c} - \frac{(\ell-1)(\ell+2)}{r^2} k \right]$$

where R is the Gaussian curvature of M^2 defined by

$$\left. \begin{aligned} R^A{}_{BCD} &= R [\delta^A{}_C g_{BD} - \delta^A{}_D g_{BC}] \\ W_{AIBC} - W_{AICB} &= W_D R^D{}_{ABC} \end{aligned} \right\} \text{ on } M^2$$

and where G_c^c and G_a^a refer to the Einstein tensor of the spherically symmetric background

$$8\pi T_{AB} = G_{AB} \equiv -2(v_{AB} + v_A v_B) + g_{AB}(2v_c{}^c + 3v_c v^c - R)$$

$$8\pi \frac{1}{2} t_a^a = \frac{1}{2} G_a^a \equiv v_c{}^c + v_c v^c - R$$

Comment: It is possible to remove all 2nd derivatives in the T_{AB} equation on the previous page by using the identity

$$\left[k_{ABIC} - k_{ACIB} - k_{BCIA} \right]^{IC} + k_c{}^c{}_{IAIB} + g_{AB} (k_{CD}{}^{ICID} - k_c{}^c{}_{ID}) = R (k_c{}^c g_{AB} - 2k_{AB}) \quad \text{on } M^2$$

One obtains

$$\begin{aligned} -16\pi T_{AB} &= 2v^c [k_{ABIC} - k_{CAIB} - k_{CBIA}] - \left[\frac{\ell(\ell+2)}{r^2} + G_c^c + G_a^a \right] k_{AB} \\ &- 2g_{AB} v^c (k_{DEIC} - k_{CDIE} - k_{CEID}) g^{ED} + g_{AB} (2v^{CID} + 4v^c v^D - G^{CD}) k_{CD} \\ &+ g_{AB} \left[\frac{\ell(\ell+1)}{r^2} + \frac{1}{2} (G_c^c + G_a^a) \right] k_D{}^D + R [k_c{}^c g_{AB} - 2k_{AB}] \\ &+ 2(v_A k_{,B} + v_B k_{,A} + k_{,AB}) \\ &- g_{AB} (2k_{,c}{}^{IC} + 6v^c k_{,c} - \frac{(\ell-1)(\ell+2)}{r^2} k) \end{aligned}$$

Problem 5: (cont'd)

These gauge invariant linearized field equations are not independent. The linearized conservation equations $\Delta(T_{\mu}^{\nu}; \nu) = 0$ have to be obeyed. In terms of the $h_{\mu\nu}$ characterized even parity perturbations one obtains a scalar and a vector equation on M^2 :

$$\tau^{-2} (r^2 T^A)_{;A} + T^1 + [1 - \ell(\ell+1)] T^2 / r^2 = \frac{1}{2} t_a^a (k - \frac{1}{2} k_c) + \frac{1}{2} t^{AB} k_{AB}$$

(Scalar)

$$\tau^{-2} (r^2 T_{AB})^{;B} - T_A \frac{\ell(\ell+1)}{r^2} - 2v_A T^1 + v_A T^2 \frac{\ell(\ell+1)}{r^2} =$$
$$= \frac{1}{2} k_{BC|A} t^{BC} + k_{CB}^{;B} t^A - \frac{1}{2} k_{c|B} t^B - k_{,c} t^c +$$
$$+ (\frac{1}{2} k_{,A} - k v_A) t_a^a + 2v^B k_{BC} t^c_A + k^B_c t^c_{A|B}$$

(Vector)

Problem 6: Derive the odd parity gauge invariant linearized Einstein field eq'ns on a spherically symmetric background from a variational principle.

suggestion: possible candidates for Lagrangians can be found in

G.W. Gibbons and M.J. Perry:
"Quantizing Gravitational Instantons"
Nucl. Phys. B146 90-108 (1978)

R.A. Isaacson: "Gravitational Radiation in the High Frequency Limit of I. The Linear Approximation and Geometrical Optics", Phys Rev 166, 1263-1271 (1968)

Problem 7: Derive the even parity gauge invariant linearized Einstein field eq'ns on a generic spherically symmetric background from a variational principle.