

Lecture 43

I) Infinitesimal coordinate transformations applied to geometrical objects (tensor, vector, and scalar fields)

II) Scalar, Vector, and Tensor harmonics on the unit two-sphere S^2 as basis functions for perturbations on a generic spherically symmetric spacetime.

I) Infinitesimal Coordinate Transformations (II)

(Perturbations in the background coordinate system.)

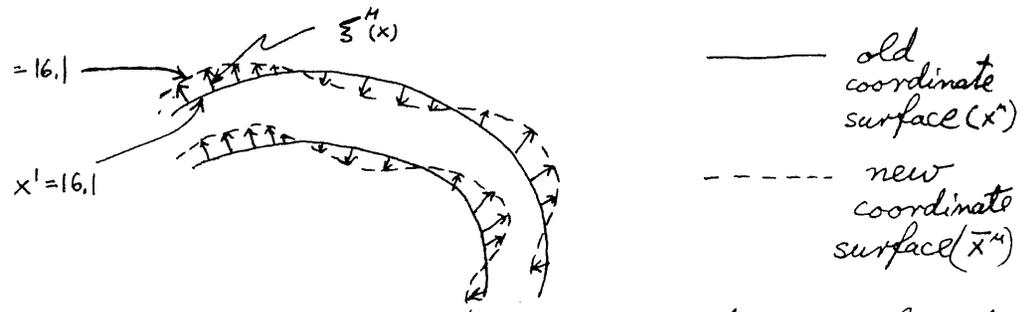
Not all perturbations in the metric coefficient functions are due to perturbations in the metric itself. Some perturbations in the coefficients ($g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x)$) are due to mere perturbational changes in the coordinate system used. From the point of view of describing phenomena involving perturbations in the gravitational field, as well as in other fields, it is clear that changes due to a mere recoordination are of less interest than changes due to actual perturbations in the tensor fields.

To see how this works in practice consider

an infinitesimal coordinate change

$$x^\mu \rightarrow x^\mu + \xi^\mu(x) \equiv \bar{x}^\mu$$

where $\xi^\mu(x)$ are the components of an infinitesimal vector field



Evidently, such a coordinate transformation is described by $\xi^\mu(x)$. The inverse transformation is

$$\begin{aligned} \bar{x}^\mu &\rightarrow x^\mu = \bar{x}^\mu - \xi^\mu(x) \\ &= \bar{x}^\mu - \xi^\mu(\bar{x} - \xi) \\ &= \bar{x}^\mu - \underbrace{\xi^\mu(\bar{x})}_{1^{st} \text{ order}} + \underbrace{\xi^\mu_{,\sigma}(\bar{x}) \xi^\sigma}_{2^{nd} \text{ order}} + \dots \end{aligned}$$

Thus $x^\mu \approx \bar{x}^\mu - \xi^\mu(\bar{x})$ where we neglected 2nd and higher order

terms. Thus we have

$$\begin{array}{|l} \bar{x}^{\mu} = x^{\mu} + \xi^{\mu}(x) \\ \bar{x}^{\mu} = \bar{x}^{\mu} - \xi^{\mu}(\bar{x}) \end{array}$$

Transformation

Inverse transformation

where ξ^{μ} are the components of a vector.

Effect of Infinitesimal Coordinate Transformation on Metric.

The effect is easy to determine. One has

$$\begin{aligned} g_{\mu\nu}(x) dx^{\mu} \otimes dx^{\nu} &= g_{\mu\nu}(\bar{x}^{\sigma} - \xi^{\sigma}(\bar{x})) d(\bar{x}^{\mu} - \xi^{\mu}(\bar{x})) \otimes d(\bar{x}^{\nu} - \xi^{\nu}(\bar{x})) \\ &= g_{\mu\nu}(\bar{x}) d\bar{x}^{\mu} \otimes d\bar{x}^{\nu} - g_{\mu\nu,\sigma} \xi^{\sigma} d\bar{x}^{\mu} \otimes d\bar{x}^{\nu} \\ &\quad - g_{\mu\nu} \xi^{\mu}_{,\sigma} d\bar{x}^{\sigma} \otimes d\bar{x}^{\nu} - g_{\mu\nu} \xi^{\nu}_{,\sigma} d\bar{x}^{\mu} \otimes d\bar{x}^{\sigma} \\ &\quad + 2^{\text{nd}} \text{ and higher order terms} \end{aligned}$$

It follows that the perturbation induced by ξ^{μ} is

$$\begin{aligned} \Delta g_{\mu\nu} &= -g_{\mu\nu,\sigma} \xi^{\sigma} - g_{\sigma\nu} \xi^{\sigma}_{,\mu} - g_{\mu\sigma} \xi^{\sigma}_{,\nu} \\ &= -(\xi_{\mu;\nu} + \xi_{\nu;\mu}) \end{aligned}$$

where comma denotes partial and semicolon covariant derivation

⊙ Problem: verify that

$$g_{\mu\nu,\sigma} \xi^{\sigma} + g_{\sigma\nu} \xi^{\sigma}_{,\mu} + g_{\mu\sigma} \xi^{\sigma}_{,\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

Conclusion:

Given an infinitesimal coordinate transformation $\bar{x}^{\mu} = x^{\mu} + \xi^{\mu}(x)$ characterized by the (infinitesimal) vector $\xi^{\mu} \frac{\partial}{\partial x^{\mu}}$, then the metric ~~will transform~~ ^{tensor} coefficient field will be transformed as follows:

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x) + h_{\mu\nu}(x)$$

where $h_{\mu\nu}(x) = -(\xi_{\mu;\nu} + \xi_{\nu;\mu})$

Comment:

If $g_{\mu\nu}(x)$ is a solution to the full Einstein field equation, then so is

$$\bar{g}_{\mu\nu} = g_{\mu\nu} - \xi_{\mu;\nu} - \xi_{\nu;\mu}$$

But $\bar{g}_{\mu\nu}$ as well as $g_{\mu\nu}$ refer to the same metric. They are different representations

of the same metric.

Comment: Suppose one has two perturbations tensor:

of the same background (unperturbed) metric, ^(say $g_{\mu\nu}$) $h_{\mu\nu}^1$ and $h_{\mu\nu}^2$. Further, suppose that

$$h_{\mu\nu}^2(x) = h_{\mu\nu}^1(x) - \xi_{\mu;\nu}^\sigma - \xi_{\nu;\mu}^\sigma$$

Then it is clear that $h_{\mu\nu}^1$ and $h_{\mu\nu}^2$

express the same perturbation of the metric, $g_{\mu\nu}(x)$.

Definition: The transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \xi_{\mu;\nu}^\sigma - \xi_{\nu;\mu}^\sigma$$

applied to the perturbation $h_{\mu\nu}$ and induced by the vector field ξ_μ is called a gauge transformation.

The metric is not the only tensor field affected by infinitesimal coordinate transformations. There are general tensor fields, say $t_{\mu\nu}(x)$; general covector fields, say $v^\mu(x)$; and scalar fields, say $\phi(x)$.

(15) effect of infinitesimal coordinate transformations on tensor, vector and scalar fields. (16)

Given: $\bar{x}^\mu = x^\mu + \xi^\mu(x)$, an inf.^l coord. transformation.

$$\begin{aligned} t_{\mu\nu}(x) dx^\mu \otimes dx^\nu &= t_{\mu\nu}(\bar{x} - \xi) d(\bar{x}^\mu - \xi^\mu) \otimes d(\bar{x}^\nu - \xi^\nu) = \\ &= t_{\mu\nu}(\bar{x}) d\bar{x}^\mu \otimes d\bar{x}^\nu - t_{\mu\nu,\sigma} \xi^\sigma d\bar{x}^\mu \otimes d\bar{x}^\nu \\ &\quad - \frac{1}{2} t_{\mu\nu,\sigma} \xi^\sigma d\bar{x}^\sigma \otimes d\bar{x}^\nu - t_{\mu\nu} \xi^\sigma_{,\nu} d\bar{x}^\mu \otimes d\bar{x}^\sigma \\ &\quad + \text{higher order terms.} \end{aligned}$$

Thus

$$\Delta t_{\mu\nu} = - t_{\mu\nu,\sigma} \xi^\sigma - t_{\sigma\nu} \xi^\sigma_{,\mu} - t_{\mu\sigma} \xi^\sigma_{,\nu}$$

$$\Delta t_{\mu\nu} = - t_{\mu\nu;\sigma} \xi^\sigma - t_{\sigma\nu} \xi^\sigma_{;\mu} - t_{\mu\sigma} \xi^\sigma_{;\nu}$$

Problem: show that the above equality does, in fact, hold true.

covector:

$$\begin{aligned} v_\mu(x) dx^\mu &= v_\mu(\bar{x} - \xi) d(\bar{x}^\mu - \xi^\mu) \\ &= v_\mu(\bar{x}) d\bar{x}^\mu - v_{\mu,\sigma} \xi^\sigma d\bar{x}^\mu - v_\mu \xi^\sigma_{,\sigma} dx^\sigma \end{aligned}$$

thus

$$\Delta v_\mu = - v_{\mu,\sigma} \xi^\sigma - v_\sigma \xi^\sigma_{,\mu}$$

$$\Delta v_\mu = - (v_{\mu;\sigma} \xi^\sigma + v_\sigma \xi^\sigma_{;\mu})$$

Problem: show that the above equality does, in fact, hold true

vector:

$$u^{\mu}(x) \frac{\partial}{\partial x^{\mu}} = u^{\mu}(\bar{x} - \xi) \frac{\partial \bar{x}^{\sigma}}{\partial x^{\mu}} \frac{\partial}{\partial \bar{x}^{\sigma}}$$

$$= u^{\mu}(\bar{x}^{\nu} - \xi^{\nu}) \left(\delta_{\mu}^{\sigma} + \xi^{\sigma}_{,\mu} \right) \frac{\partial}{\partial \bar{x}^{\sigma}}$$

$$= u^{\mu} \frac{\partial}{\partial \bar{x}^{\mu}} - u^{\mu}_{,\nu} \xi^{\nu} \frac{\partial}{\partial \bar{x}^{\mu}} + u^{\mu} \xi^{\sigma}_{,\mu} \frac{\partial}{\partial \bar{x}^{\sigma}}$$

thus

$$\Delta u^{\mu} = - \left(u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma} \right)$$

THIS WILL BE RECOGNIZED AS THE COMPONENTS OF $-\mathbb{L}_{\xi} u^{\mu}$, the commutator of u and ξ !

$$\Delta u^{\mu} = - \left(u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma} \right)$$

Problem: show that the equality

$$u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma} = u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma}$$

does in fact hold true.

scalar:

$$\begin{aligned} \phi(x) &= \phi(\bar{x} - \xi) \\ &= \phi(\bar{x}) - \phi_{;\sigma} \xi^{\sigma} \end{aligned}$$

thus

$$\Delta \phi = - \phi_{;\sigma} \xi^{\sigma}$$

Thus the change here is merely the directional derivative.

(17)

Definition: Consider perturbations

$\Delta h_{\mu\nu}, \Delta t_{\mu\nu}, \Delta v_{\mu}, \Delta u^{\mu}, \Delta \phi$
in the geometrical objects

$g_{\mu\nu}, t_{\mu\nu}, v_{\mu}, u^{\mu}, \phi$.

Let ξ^{μ} induce an infinitesimal coordinate transformation in these background geometrical objects.

The following transformations are called gauge transformations.

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - (\xi_{\mu;\nu} + \xi_{\nu;\mu})$$

$$\Delta t_{\mu\nu} \rightarrow \Delta t_{\mu\nu} - (t_{\mu\nu;\sigma} \xi^{\sigma} + t_{\sigma\nu} \xi^{\sigma}_{;\mu} + t_{\mu\sigma} \xi^{\sigma}_{;\nu})$$

$$\Delta v_{\mu} \rightarrow \Delta v_{\mu} - (v_{\mu;\sigma} \xi^{\sigma} + v_{\sigma} \xi^{\sigma}_{;\mu})$$

$$\Delta u^{\mu} \rightarrow \Delta u^{\mu} - (u^{\mu}_{;\sigma} \xi^{\sigma} - u^{\sigma} \xi^{\mu}_{;\sigma})$$

$$\Delta \phi \rightarrow \Delta \phi - (\phi_{;\sigma} \xi^{\sigma})$$

Comment: The expressions in the parenthesis on the r.h.s., "the gauge changes", are the Lie derivatives of the geometrical objects.

Notation:

the derivative w.r.t. ξ^{μ} of $t_{\mu\nu}$

$$\mathbb{L}_{\xi} t_{\mu\nu} = t_{\mu\nu;\sigma} \xi^{\sigma} + t_{\sigma\nu} \xi^{\sigma}_{;\mu} + t_{\mu\sigma} \xi^{\sigma}_{;\nu}$$

similarly for others.

(18)

II)

Nearly Spherically Symmetric Configurations.

Scalar, Vector, and Tensor Harmonics
on the unit two sphere.

It is the existence of highly symmetric configurations, such as spherically symmetric ones, that makes possible the existence of many dramatic and fascinating phenomena in Nature. They are deviations away from spherical symmetry, waves generated and propagating on a spherically symmetric background.

Examples of such waves are hydrodynamical waves, as well as electromagnetic, ^{and} gravitational waves evolving in a static or evidently time dependent spherically symmetric background.

The waves are quite general. They can even be static perturbations, but their common denominator, we shall assume, is that they are linear, i.e. satisfy the linear

superposition principle

For a spherically symmetric configuration the four dimensional spacetime and hydrodynamical background is provided by tensor fields which reduce to equivalent geometrical objects on the two dimensional reduced (= "quotient") space $M^2 = M^4/S^2$.

Examples are

On M^4 scalar field: p ↔ On M^2 $p(x^a)$ scalar field

covector field: v_μ ↔ $v_B(x^a)$ covector field

tensor field: $t_{\mu\nu}$ ↔ $\begin{cases} t_{AB}(x^a) & \text{tensor field} \\ \frac{1}{2} t^a_a & \text{scalar field} \end{cases}$

metric tensor: $g_{\mu\nu}$ ↔ $\begin{cases} g_{AB}(x^a) & \text{metric tensor} \\ r(x^a) & \text{scalar field} \end{cases}$

These quantities are governed by the reduced Einstein field equations and the correspondingly implied Euler equations of hydrodynamics on M^2 . See Lecture 25.

A non-spherical configuration, for example a star suffering non-spherical gravitational collapse and the concomitant interactive evolution of hydrodynamical and gravitational waves, falls into the same geometrical framework based on $M^2 = M^4/S$. The only proviso is that the deviations away from spherical symmetry are small enough so that they obey the linear superposition principle.

Because of the spherical symmetry of the background, these deviations can be expanded in terms of spherical harmonics with expansion coefficients on M^2 . Indeed we have the following

Proposition

Any scalar, vector, or tensor field on a spherically symmetric manifold M^4 induces a corresponding unique set of geometrical objects on M^2 .

a. For a scalar field, say $\Delta p(x^0, x^i, \theta, \varphi)$ on M^4 we consider its expansion in terms of spherical scalar harmonics $Y^{lm}(\theta, \varphi)$

$$\Delta p(x^0, x^i, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \Delta P_{lm}(x^0, x^i) Y^{lm}(\theta, \varphi)$$

This implies

$$\boxed{\Delta p \text{ on } M^4} \longleftrightarrow \boxed{\Delta P_{lm}(x^0, x^i) \text{ on } M^2} \quad -l \leq m \leq l, l=0,1,2,\dots$$

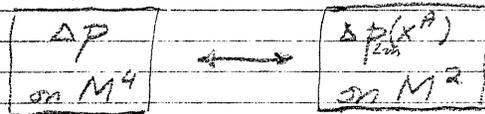
Thus, corresponding to the scalar field $\Delta p(x^0, x^i, \theta, \varphi)$ on M^4 there is a set of scalar fields $\Delta P_{lm}(x^0, x^i)$ on M^2 . This correspondence is one-to-one.

Notation:

For the purpose of typographical efficiency we shall (i) suppress reference to the angular integers m and l , and (ii) omit the summation sign in the expansion whenever such suppression and omission appears convenient. Thus the correspondence for scalar fields will also be written as

$$\Delta P = \Delta P_{\ell m}(x^A) Y$$

so that

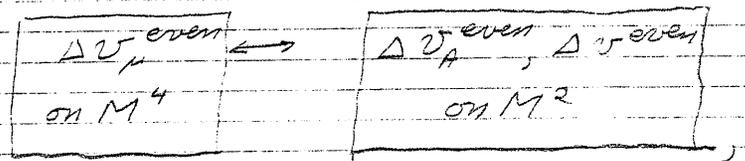


b) For a covector field, say $\Delta v_{\mu}^{(A, B, P)}$ on M^4

there is an "even" part

$$[\Delta v_{\mu}^{\text{even}}] = \begin{bmatrix} \Delta v_{\mu}^{\text{even}} & Y(\theta, \varphi) \\ \Delta v_{\alpha}^{\text{even}} & Y_{, \alpha}(\theta, \varphi) \end{bmatrix}$$

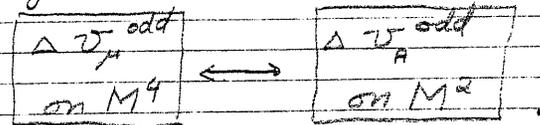
which yields



and an "odd" part

$$[\Delta v_{\mu}^{\text{odd}}] = \begin{bmatrix} 0 \\ \Delta v^{\text{odd}} S_{\alpha}(\theta, \varphi) \end{bmatrix}$$

which yields



Comments:

- (i) The "even" and the "odd" parts, whose sum comprises the total covector field,

$$\Delta V_{\mu} = \Delta V_{\mu}^{\text{even}} + \Delta V_{\mu}^{\text{odd}}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^l \begin{bmatrix} \Delta V_{A_{2m}}^{\text{even}} \\ 0 \\ 0 \end{bmatrix} Y_{(p,q)}^{2m} + \sum_{l=1}^{\infty} \sum_{m=1}^l \Delta V_{B_{2m}}^{\text{even}} \begin{bmatrix} 0 \\ 0 \\ Y_{ja}^{2m} \end{bmatrix} + \sum_{l=1}^{\infty} \sum_{m=1}^l \Delta V_{C_{2m}}^{\text{odd}} \begin{bmatrix} 0 \\ 0 \\ S_a^{2m} \end{bmatrix}$$

are obtained from two entirely different sets of vector harmonics on the unit two sphere:

"even" harmonics = $\{ Y_{ja}^{2m} : -l \leq m \leq l, l = 1, 2, \dots \}$

"odd" harmonics = $\{ S_a^{2m} \equiv Y_{jc}^{2m} \gamma^{cb} \epsilon_{ba} : -l \leq m \leq l, l = 1, 2, \dots \}$

The "even" vector harmonics are merely the gradient of the spherical harmonics

$$\{ Y_{ja}^{2m} \} = \left\{ \frac{\partial Y^{2m}}{\partial \theta}, \frac{\partial Y^{2m}}{\partial \phi} \right\}$$

By contrast the "odd" vector harmonics are the curl of this gradient. Indeed, using

$$[\gamma^{cb}] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2 \theta} \end{bmatrix}$$

$$\sqrt{\det[b,a]} = [\epsilon_{ba}] = \begin{bmatrix} 0 & \epsilon_{\phi\theta} \\ -\epsilon_{\phi\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sin \theta \\ -\sin \theta & 0 \end{bmatrix} =$$

we have

$$\{ S_{\theta}^{2m}, S_{\phi}^{2m} \} = \left\{ -\frac{\partial Y^{2m}}{\partial \phi} \frac{1}{\sin \theta}, \frac{\partial Y^{2m}}{\partial \theta} \sin \theta \right\}$$

(ii) Like the curl in three dimensions, these "odd" vector harmonics have vanishing divergence on the unit two sphere

$$S^a : a = 0 \text{ because } (Y_{jc} \gamma^{cb} \epsilon_{ba})^{;a} = Y_{jc;a} \epsilon^{ca}$$

Here the colon ";" refers to the covariant derivative on the two sphere, and we used the consequent covariant constancy of γ and ϵ :

$$\gamma^{cb};d = 0, \epsilon_{ba};d = 0$$

By contrast, the divergence of Y_{ja} is $Y_{ja}^{;a} = -l(l+1)Y$

(iii) Another difference between the "even" and the "odd" vector harmonics is that they transform differently under the parity operation

$$P: (\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi),$$

i.e. under reflection diagonally

across the two-sphere:

$$P Y_{lm}(\theta, \varphi) = (-1)^l Y_{lm}(\theta, \varphi)$$

$$P Y_{l,0}^{lm}(\theta, \varphi) dx^a = (-1)^l Y_{l,0}^{lm}(\theta, \varphi) dx^a \text{ "even" harmonics}$$

$$P S_{l,0}^{lm}(\theta, \varphi) dx^a = (-1)^{l+1} S_{l,0}^{lm}(\theta, \varphi) dx^a \text{ "odd" harmonics}$$

As one can see the terms "even" and "odd" harmonics are misnomers. One actually should talk of $(-1)^l$ parity and $(-1)^{l+1}$ parity vector harmonics.

(iv) Finally we note that the $(-1)^l$ parity vector harmonics are orthogonal to the $(-1)^{l+1}$ parity vector harmonics

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi Y_{l,0}^{lm} S_{l,0}^{lm'} \gamma^{ab} \sin\theta d\theta d\varphi \\ &= \int_0^{2\pi} \int_0^\pi Y_{l,0}^{lm} S_{l,0}^{lm'} \sin\theta d\theta d\varphi = 0 \end{aligned}$$