Lecture 0

How Newton was led to his universal law of gravitation: a road map

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Abstract

We identify the roots and the fundamental premise of Newton's scientific achievements: to grasp the nature of the world, one's thinking must begin with information received from the world. Adopting it, we apply elementary calculus to three pieces of information, Kepler's three laws, to obtain Newton's universal law of gravitation.

1 Newton's fundamental premise and its origin

1.1 The fundamental premise

Q: What was the fundamental premise which paved the way towards Newton's unprecedented achievement?
Why was he successful, while others (like Descarte) were not?

A: Newton stated it thusly:

> $\cdots$ I frame no hypotheses; $\cdots$
> The word "hypothesis" is here used by me to signify only
> (i) such a proposition as is not a phenomenon
> (ii) nor deduced from any phenomena,

but assumed or supposed – without any experimental proof [whatsoever].

To be more explicit[1], he used "hypothesis" to refer to an *arbitrary* statement, i.e. a claim unsupported by any observational *evidence*. Here are some examples:

(i) The works of Plato are being studied by a reading group of gremlins on the planet Venus (to pick an obvious example).

(ii) Colored light is produced by rotating particles and white light is less produced by nonrotating particles (Descarte).

(iii) White light is a symmetrical wave pulse (Robert Hooke).

(iv) Quarks are composed of strings in a 26-dimensional space (20th century string theorist), etc.[2]

As Wolfgang Pauli would say, none of these statements is right; they aren't even wrong.
Following Newton, what Pauli was directing attention to was that there are three types of claims:

1. Right ones, which are true because they have a positive relationship to reality,

2. false ones, which are untrue because they have a negative relation to reality, and

3. arbitrary ones, for which there is no evidence whatsoever: they are detached from reality.

and it is the arbitrary ones "which aren't even wrong".

Q: What cognitive value, if any, did Newton see in one's contemplation of arbitrary claims?

[1]Newton did not mean to reject out of hand all hypotheses that lacked full experimental *proof*.

[2]A continuation of this list would include astrology, intelligent design, clairvoyance, ESP, God, an afterlife, reincarnation, and other misintegrations.

A: Newton must be credited with being the first one to identify what in 20th century vernacular is called *Garbage In Gargage Out (G.I.G.O.).* Writing to a friend, he said:

> "If anyone may offer conjectures about the truth of things from the mere possibility of hypotheses, I do not see by what stipulation anything certain can be determined in any science; since one or another set of hypotheses may always be devised which will appear to supply new difficulties. Hence I judged that one should *abstain from contemplating hypotheses, as [one does] from improper argumentation.*"

In other words, one's thinking ("contemplation") should not start with *Garbage,* i.e. arbitrary claims ("hypotheses") because the output, "conjectures about the truth of things," will also be *Garbage,* just as one gets "from improper argumentation" .
Furthermore, as David Harriman puts it[3] , one cannot even achieve the misguided goal of disproving an arbitrary idea. Such a claim can always be shielded by further arbitrary assertions ("one or another set of hypotheses") There is only one way out of such a proliferating web of arbitrary conjectures, and that is to dismiss them outright as uncognitive and unworthy of attention.

This is why Newton insisted that the arbitrary be rejected *without contemplation.*

With this Newton introduced a new epistemological principle into the theory of knowledge:
The outright dismissal of arbitrary claims, without contemplation!
For this principle alone Newton deserves to be regarded as the greatest epistemologist of his era.

Q: What, then, is Newton's fundamental premise stated positively?

A: To grasp *the nature of the world* one's thinking has to start with *information received from the world.*

[3] *THE LOGICAL LEAP: Induction in Physics*, by D. Harriman, With an Introduction by L. Peikoff (New American Library, New York, N.Y., 2010), page 65.

a) *What* is *the nature of the world?* The world is a causal realm ruled by natural law. It is not a realm of inexplicable miracles ruled by a supernatural power, nor is it an unintelligible chaos ruled by chance. Instead, *the nature of the world* is expressed by the observation that

> "Things are what they are because they were what they were, and things will be what they will be because they are what they are"

This is the law of causality, Aristotle's *law of identity* (everything has a specific nature; things are what they are; A is A)
applied to actions.

b) That one's thinking "start with information received from the world" is the starting point of Aristotle's epistemology, *the evidence of the senses,*

be they aided or unaided by specialized instruments.

1.2 Its origin

Newton did not arrive at his fundamental premise in a cultural vacuum.

Q: What was the frame of reference – the context – that led to the achievements of Newton, those before him, and those after him?

A: Here it is essential to realize that Aristotle, whose works Newton had studied as a college student, may be considered as the cultural barometer of Western History.
Whenever his influence dominated the scene it paved the way towards histories most brilliant eras, whenever it fell so did mankind.

Aristotle's revival in the 13th century brought men to the Renaissance, and the Renaissance led to the Age of Reason, the Enlightenment. Indeed, Galileo was born in the year that Michelangelo died (1564), and Newton was born on the day that Galileo died (1642).

2 Newton's universal law of gravitation

The Enlightenment was ushered in by Newton's unprecedented achievements. There were three of them:

1. His *Opticks*, an inspiration and exemplar of Induction and the Experimental Method.

2. His infinitesimal calculus.

3. His universal law of gravitation.

All three of them illustrate Aristotle's dictum which Newton adopted as his basic premise:

> "To grasp the nature of the world one's thinking has to start with information received from the world".

To grasp the nature of gravitation, Newton's thinking started with information about the dynamics of moving bodies,

$$m \times \overrightarrow{acceleration} = \overrightarrow{Force},$$

applied to the motion of planets as given by Kepler's three laws.

2.1 Kepler's three laws

(1) The radius vector sun-planet sweeps out equal areas in equal times.

(2) The trajectory of each planet is an ellipse with the sun located at one focus,

$$r = \frac{p}{1 - \epsilon \cos\theta} \; .$$

(The parameter p is called the semi-latus rectum of the ellipse. It is the vertical distance from the focus to ellipse. The parameter ϵ is the eccentricity of the ellipse.)

(3) The square of the planets' orbital periods vary as the third power of the major axes of their ellipses:

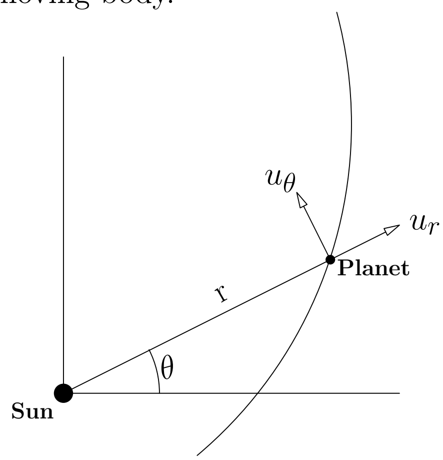
$$\frac{T^2}{a^3} = same\ const.\ for\ all\ planets\ .$$

2.2 Newton's first step: acceleration of a moving body

Using his second law of motion,

$$m\frac{d^2\overrightarrow{R}}{dt^2} = \overrightarrow{F},$$

Newton determined $\overrightarrow{F}$ by evaluating the acceleration along the trajectory of a moving body.



a) Location: $\overrightarrow{R} = r\overrightarrow{u_r}$;

$$\begin{aligned} \overrightarrow{u_r} &= \overrightarrow{i}\cos\theta &+ \overrightarrow{j}\sin\theta \\ \overrightarrow{u_\theta} &= -\overrightarrow{i}\sin\theta &+ \overrightarrow{j}\cos\theta \end{aligned}$$

b) Velocity:

$$\begin{aligned} \tfrac{d\overrightarrow{R}}{dt} &= \tfrac{dr}{dt}\overrightarrow{u_r} + r\tfrac{d\overrightarrow{u_r}}{dt} \\ &= \tfrac{dr}{dt}\overrightarrow{u_r} + r\overrightarrow{u_\theta}\tfrac{d\theta}{dt} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \overrightarrow{u_\theta} &= \tfrac{d\overrightarrow{u_r}}{d\theta} \\ \tfrac{d\overrightarrow{u_\theta}}{d\theta} &= -\overrightarrow{u_r} \end{aligned}$$

c) Acceleration:

$$\frac{d^2\overrightarrow{R}}{dt^2} = \frac{d^2r}{dt^2}\overrightarrow{u_r} + \overbrace{2\frac{dr}{dt}\overrightarrow{u_\theta}}^{coriolis\,acc'n} \overbrace{-r\overrightarrow{u_r}\left(\frac{d\theta}{dt}\right)^2}^{centripetal\,acc'n} + r\overrightarrow{u_\theta}\frac{d^2\theta}{dt^2} \tag{1}$$

$$= \underbrace{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]}_{a_r}\overrightarrow{u_r} + \underbrace{\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)}_{a_\theta}\overrightarrow{u_\theta} \tag{2}$$

2.3 Newton's second step: use Kepler's laws

Applying $\overrightarrow{F} = m\overrightarrow{a}$ to Kepler's three laws yields both the direction and the magnitude of the gravitational force on a planet.

2.3.1 Kepler's first law

Equal areas in equal times, $\Delta(\text{area}) \propto \Delta t$, implies $\frac{d(\text{area})}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{c}{2}$. Consequently,

$$a_\theta = 0.$$

Thus the acceleration, Eq.(2), is *purely radial* along the direction sun-planet:

$$\frac{d^2\overrightarrow{R}}{dt^2} = a_r\overrightarrow{u_r}.$$

This is the first key result.

2.3.2 Kepler's second law

Next consider one of Kepler's ellipses having semi-latus rectum p and eccentricity ϵ. Calculate the radial acceleration a_r using Kepler's first and second laws:

$$\begin{array}{lll}
r & = \frac{p}{1-\epsilon\cos\theta} & \Leftarrow \mathit{Kepler}\ (2) \\
\frac{dr}{dt} & = -\frac{p\epsilon\sin\theta}{(1-\epsilon\cos\theta)^2}\frac{d\theta}{dt} & \\
 & = -\frac{\epsilon\sin\theta}{p}\underbrace{r^2\frac{d\theta}{dt}}_{c} & \Leftarrow \mathit{Kepler}\ (1) \\
 & = -\frac{\epsilon\sin\theta}{p}c & \Leftarrow \mathit{Kepler}\ (1) \\
\frac{d^2r}{dt^2} & = -\frac{c}{p}\epsilon\cos\theta\frac{d\theta}{dt} & \\
 & = \frac{c}{p}\left(\frac{p}{r}-1\right)\frac{d\theta}{dt} & \Leftarrow -\epsilon\cos = \frac{p}{r}-1 \quad \Leftarrow \mathit{Kepler}\ (2) \\
 & = \frac{c}{p}\left(\frac{p}{r}-1\right)\frac{c}{r^2} & \Leftarrow \frac{d\theta}{dt} = \frac{c}{r^2} \qquad\quad \Leftarrow \mathit{Kepler}\ (1) \\
r\left(\frac{d\theta}{dt}\right)^2 & = \frac{c^2}{r^3} & \Leftarrow \mathit{Kepler}\ (1)
\end{array}$$

Subtracting the last two lines, one finds that the radial acceleration a_r in Eq.(2) is

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{c^2}{p}\frac{1}{r^2}\ .$$

Thus, not only is the acceleration, Eq.(2), of a moving planet *purely radial*, but its *magnitude* is inversely proportional to its squared distance, with a constant of proportionality constant (c^2/p) that depends on the square of the planet's areal velocity and the shape of the planetary ellipse.

Question: Is this acceleration the same for all planets?
The answer depends on the orbital periods of the planets.

2.3.3 Kepler's third law

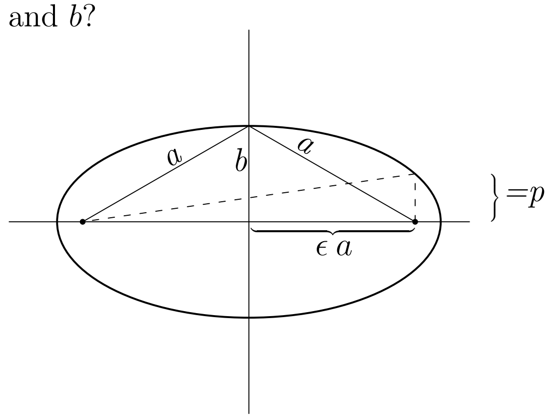
Kepler's first law implies that the orbital period is proportional to the planetary ellipse. This ellipse has major and minor axes a and b. Consequently,

$$\frac{c}{2} = \frac{d(\text{area})}{dt} \Rightarrow \frac{c}{2}T = \text{area} = \pi ab$$

Hence

$$\boxed{c = \frac{2\pi ab}{T}}$$

Q: What is the relation between the semi-latus rectum p and the two axes a and b?



A: Passing through the two foci of the ellipse are its two lati recti ("straight sides"), the vertical chords through the two focal points located at $\pm\epsilon a = \pm\sqrt{a^2 - b^2}$. The size of each latus rectum is $2p$. Thus one has a right triangle whose two sides are $2\epsilon a$ and p, and whose hypotenuse is $2a - p$. Pythagoras tells us that

$$p^2 + (2\epsilon a)^2 = (2a - p)^2.$$

Using $(\epsilon a)^2 = a^2 - b^2$one finds that the semi-latus rectum is

$$\boxed{p = \frac{b^2}{a}\ .}$$

By applying the two boxed expressions to the radial acceleration a_r

$$\begin{aligned} a_r &= -\frac{c^2}{p}\frac{1}{r^2} = -\left(\frac{2\pi ab}{T}\right)^2 \frac{a}{b^2}\frac{1}{r^2} \\ &= -4\pi^2\frac{a^3}{T^2}\frac{1}{r^2}\ . \end{aligned}$$

Using Kepler's third law, one obtains

$$\boxed{a_r = -\frac{\gamma}{r^2}}$$

where $\gamma = \gamma(M)$ is a constant which is the same for all planets, but which depends on the mass M of of the sun in an as-yet-unspecified way.

2.4 Newton's third step: use his 2nd and 3rd law of motion.

Q: What is the value of that planet-independent constant γ?
Newton answers this question by resorting to his second and third law of motion.

(i) Applying his second law to a planet of mass m,

$$m \times \overrightarrow{(\text{acceleration})} = \vec{F}\ ,$$

one obtains the purely redial gravitational force,

$$\boxed{F_r^{SP} = -\frac{m\gamma(M)}{r^2}}$$

acting on the planet. This is the force with which the sun attracts the planet.

(ii) On the other hand, by his third law there is an equal but opposite force acting on the *sun,*

$$F_r^{PS} = -F_r^{SP} \;,$$

which is to say that the weight of the planet towards the sun is equal to the weight of the sun towards the planet. Consequently,

$$\frac{M\,\Gamma(m)}{r^2} = \frac{m\,\gamma(M)}{r^2} \;.$$

This equality holds for all pairs of masses m and M. Consequently,

$$\boxed{F_r = -\kappa \frac{Mm}{r^2} \;.}$$

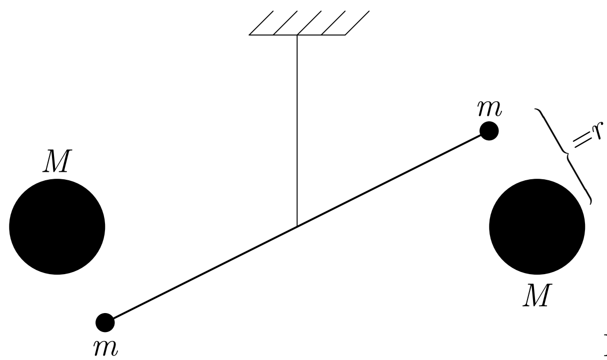
Here κ, *Newton's gravitational constant*, is a universal constant independent of M and m. The boxed equation is a mathematical statement of Newton's universal law of gravitation.

2.5 The Cavendish Experiment (1789)

The universal constant κ has the value

$$\kappa = \frac{1}{15\,000\,000} \left[\frac{cm^3}{gr\ sec^3}\right]$$

in c.g.s. units. This constant is determined by measuring the attractive force between masses separated by a known distant r. One suspends two small masses m from a torsional balance.



By bringing large masses M to each of the masses m, Cavendish measured the gravitational force F_r by measuring the angular deflection of the pendulum.
Newton's third law of motion, "For every action there is an equal and opposite reaction", applied to his law of gravitation, implies that the weight of an apple attracted by the earth's gravity equals the weight of the earth attracted by the gravity of the apples. This equality determines the mass of the earth once the weight of the apple and its distance from the (center of the) earth have been measured.