

LECTURE 1

The concept "Gravitation"; Where does it come from?

A. Gravitation's observational basis

B. Mathematization of gravitation according
to

I. Galileo

II. Kepler

III. Newton

IV. Lagrange & Hamilton

V. Einstein

The theme of MATH5757 is to grasp the nature of the world, in particular the existence and the nature of "gravitation", which is the subject to support this theme.

- A.) To do this, our thinking must start with information received from the world. In the case of gravitation this information is in the specific form of the laws of motion of bodies. It is precisely in terms of the observed motion of bodies that one arrives at the concept "gravitation".

The concept "gravitation" is formed by a process of "measurement omission" [The process of concept formation is explained on pages 11-18 of Chapter 2 ("Concept Formation") in "Introduction to Objectivist Epistemology" by Ayn Rand; also summarized in the Q&A on pages 137-139]

The formation of the concept "gravitation" starts with observation of the various kinds of motions of bodies and then singling out a particular type of motion which is different from all the rest. The common feature(s) which

1.3a

particularizes this type of motion,
are the characteristic imprint(s),
the signature(!), that gravitation
imparts to the motion of bodies.

Historically what nowadays is
identified as the gravitation phenomena,
has been the chief motivating force
for the mathematization (= "mathema-
tical formulation") of the laws of
motion. This is because they are
the premier tool for identifying
what gravitation is.

It was Galileo, Kepler, Newton, Einstein & others
who, each in their own way, identified and
mathematized gravitation in their own
way.

GO TO P 1.4

I. Galileo: ("E pur si muove" → "And yet it moves") 1.4

"Independence of Horizontal and Vertical Motion."

Using the experimental method in studying the motion of a projectile, Galileo found that its horizontal and its vertical motion are independent of each other

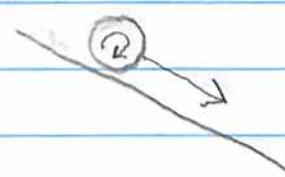
Projectile



Leaning tower
of Pisa



Inclined plane



"parabola"

"vertical acc'n
is indep. of a
body's composition"

"distance
 $\propto (time)^2$ "

1. (Horizontal) = const
 (velocity)

2. (Vertical) = constant
 (acceleration)

a) (Vertical) = $g \times \text{time}$
 (velocity)

b) (Vertical) = $\frac{1}{2} g \cdot (\text{time})^2$
 (distance)

Comments:

1. The fact that the horizontal velocity is constant is a special case of the law of inertial motion of bodies ("Newton's 1st Law of Motion")

2. Galileo identified the import of gravitation by the proportionality constant g between the vertical velocity and the time of travel of the body (= point particle)

II. Kepler:

1.5

"Kepler's 3 laws of planetary motion!"

From direct observations Kepler by inductive reasoning, mathematized the motion of planets (and moons) into his three laws.



Areal
velocity
=constant



Elliptic
orbit

3.

$$GM^2 = \omega^2 R^3$$

1-2-3 law

$$(G = \frac{1}{15000} \frac{m^3}{kg \cdot sec^2} = \frac{1}{15000000} \frac{m^3}{g \cdot sec^2})$$

1. The radius vector sun-planet sweeps out equal areas in equal times

2. Planets travel in an ellipse, with the sun at one focus,

$$3. G \left(\frac{\text{mass of the sun}}{\text{period}} \right)^2 = \left(\frac{2\pi}{\text{period}} \right)^2 \left(\frac{\text{major axis}}{\text{minor axis}} \right)^2$$

$$GM^2 = \omega^2 R^3$$

III. Newton: "Universal Law of Gravitation"

Galileo's motion of bodies +

+ Local vectorial calculus +

$$+ \vec{F} = m \frac{d^2 \vec{x}}{dt^2} +$$

+ Kepler's 3 laws of planetary motion +

+ observational data about
planets, comets, moons, cannonballs,
apples, etc

$$\Rightarrow \boxed{\text{Newton's law of gravitation: } \frac{d^2 \vec{x}}{dt^2} = -m_{\text{grav}} \frac{GM}{r^3} \vec{x}} \quad (1)$$

Exercise:

Show that

$$\left\{ \begin{array}{l} \text{Kepler's 3 laws} \\ \text{a) } \Rightarrow \\ \text{b) } \Leftarrow \end{array} \right\} \begin{array}{l} (\text{grav'l force}) = \frac{GM}{r^3} \vec{x} \\ (\text{field}) \end{array}$$

Comment 1:

"Showing the implication" \Rightarrow "entails only differential calculus (see P5-10 of Lectures). This is ^(also) done in Math 1181H and in Math 4551 here at OSU."

Comment 2:

Showing the implication " \Leftarrow " is more challenging mathematically.

This is because it entails integral calculus.

Comment 3:

By introducing his 2nd Law

$$\frac{d}{dt} \left(m \frac{d\vec{x}}{dt} \right) = \vec{F},$$

Newton accomplished several feats in one fell swoop:

(i) He introduced a new concept,

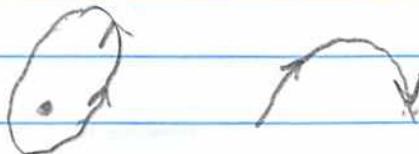
the inertial mass of a body

(ii) Whereas Kepler and Galileo

mathematized the motion in terms

of global geometrical figures

(ellipses, parabolas, etc),



Newton, having introduced the concept of mass and $\vec{F} = m\vec{a}$, did so in terms of locally defined differential equations such as the boxed one on page 1.7, or more generally

$$\boxed{m_{\text{inertial}} \frac{d^2\vec{x}}{dt^2} = -m_{\text{gravite}} G \sum_i M_i \frac{\vec{x} - \vec{x}_i(t)}{|\vec{x} - \vec{x}_i(t)|^3}} \quad (2)$$

whenever there are several gravitation forces due masses M_1 at $\vec{x}_1(t)$, M_2 at $\vec{x}_2(t)$...

(ii) He gave a local definition of acceleration by means of a double limiting process applied to differential equations.

Comment 4:

What is the difference between Newton's contribution to our understanding of the motion of bodies and that of Kepler and Galileo?

Galileo & Kepler's geometrical figures
mathematize the motion of bodies
kinematically, without any reference
to their masses.

Newton's equations for the motions of
bodies mathematize them dynamically
in terms of their masses.

GO TO Page 1.11

The time interval between Newton and Einstein (18th and 19th century) was marked by the development of the "Hamilton's Principle" of least action by Euler, Lagrange and Hamilton.

This principle uses the calculus of variations to replace Newton's vectorial equation of motion with the requirement that the scalar integral, the "action" of the mechanical system, be an extremum,

Euler : }
Lagrange : }
Hamilton : } $\delta \int S (K.E. - P.E.) dt = 0$ independent of the coord. chosen for K.E. & P.E.

The main virtue of this formulation of the classical laws of motion is that the action of a mechanical system is a scalar and that the extremum of this scalar is independent of the choice of coordinates used to describe the mechanical systems. If one reexpresses the Langrangian K.E.-P.E. in terms of different coordinates,

then the resulting Lagrange's equations of motion (whose solution extremizes the action) still describe the same mechanical system, but relative to that new set of coordinates.

Question 1: What is the physical origin of Hamilton's Principle as formulated by Lagrange.

Answer 1: The observation-based reasoning leading to this principle is given on P1-7 of the ensuing article.

"Lagrangian Mechanics and the 3-Body Problem"

Question 2: Is it possible to give a non-trivial application of this principle?

Answer 2: Yes. Pages 9-25 of the ensuing article develop the theory of the "Restricted 3-Body Problem".

V. Einstein

By the time Einstein started examining the concept "gravitation", he had at his disposal, and then made excellent use of, the highly developed art of analytical mechanics as formulated by Lagrange and Hamilton.

In 1913 he took a key step. Using Hamilton's principle of least action, he equated the Hamilton action integral to the coordinate frame independent length

$$\int d\tau = \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu}$$

of a worldline between two events,

and then pointed out that (more abstractly)

the metric tensor field

$$g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$$

is where gravitation puts its imprints,
i.e. characterizes the gravitational
field.

Thus instead of using the boxed
Eq.(1) on page 1.7 to mathematize
gravitation's imprints on the motion
of bodies, Einstein applied Hamilton's
variational
action principle, the boxed equation on
page 1.11, to

$$\delta \left\{ S \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \right\} = 0.$$

He thus obtained

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0,$$

which

(a) reduce to Newton's Eq.(2) on P.1.9

$$m_{\text{inertia}} \frac{d^2 \vec{x}}{dt^2} + m_{\text{grav}} \nabla \underbrace{\phi_{\text{grav}}(\vec{x}, t)}_{c^2} = 0$$

where $t = c\tau$

"dimensionless"

in terms of the Newtonian gravitational

potential $\phi_{\text{grav}}(\vec{x}, t)$, and

(b) express the imprints of gravitation

on the motion of bodies in the form of

geodesic states of motion on a spacetime

manifold with a metric tensor field,
and hence

(c) mathematizes gravitation in terms

of geometry.