Geometric Properties of the Particle Density-flux 3-form

$$^*S = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}.$$

One way of mathematizing matter in motion is in terms the eventwise previously defined particle current 4-vector,

$$\begin{aligned} \mathbf{S} \equiv N\mathbf{u} &= \underbrace{N(x^\nu)u^\mu}_{S^\mu} \frac{\partial}{\partial x^\mu}. \\ &= \mathbf{e}_\mu S^\mu \end{aligned} \tag{1}$$

But in order to mathematize the causal relation between matter and gravitation/geometry one also needs to know the <u>amount</u> of matter in a given generalized volume of spacetime. For any triad of spacetime vectors $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ this volume is expressed by the "volume 1-form"

$$\begin{aligned} \underset{\sim}{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C}) &= \Sigma_\mu(\mathbf{A}, \mathbf{B}, \mathbf{C}) dx^\mu \\ &\equiv \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} (\mathbf{A}, \mathbf{B}, \mathbf{C})\, dx^\mu \\ &= \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma \; dx^\mu \end{aligned}$$

or equivalently by the "volume vector"

$$\begin{aligned} \mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C}) &= \mathbf{e}_\nu g^{\nu\mu} \Sigma_\mu(\mathbf{A}, \mathbf{B}, \mathbf{C}) \\ &\equiv \mathbf{e}_\nu \Sigma^\nu(\mathbf{A}, \mathbf{B}, \mathbf{C}) \end{aligned} \tag{2}$$

Here

$$\epsilon_{\mu\alpha\beta\gamma} = \sqrt{-\det g_{\sigma\rho}}\, [\mu\alpha\beta\gamma]$$

are the components of the totally anti-symmetric Levi-Civita tensor in 4-d with metric

$$g_{\sigma\rho}\, dx^\sigma dx^\rho.$$

The importance of this "volume vector" derives from its four geometrical properties:

1. First of all, the "volume vector" $\mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C})$, Eq.(2), in 4-d spacetime is the Lorentzian version of the "area vector", i.e. the familiar cross product, in 3d Euclidean space,

$$\begin{aligned} {}^{(2)}\mathbf{\Sigma}(\vec{A}, \vec{B}) &= \vec{e}_\ell \; g^{\ell i} \; {}^{(2)}\Sigma_i(\vec{A}, \vec{B}) \\ &\equiv \vec{e}_\ell \; {}^{(2)}\Sigma^\ell(\vec{A}, \vec{B}), \end{aligned} \tag{3}$$

which as shown in the footnote[1] below is equal to $\vec{A} \times \vec{B}$.
Relative to rectilinear (x, y, z)-coordinates, for an area subtended by an as-yet-unspecified pair of vectors, that area vector is

$$^{(2)}\mathbf{\Sigma} = \vec{i}\, dy \wedge dz + \vec{j}\, dz \wedge dx + \vec{k}\, dx \wedge dy.$$

This is the mathematical building block for constructing a syrface integral in Euclidean space.

2. Secondly, the vector $\mathbf{\Sigma}$ is (Lorentz)-orthogonal to each of the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ which span span this volume:

$$\mathbf{A} \cdot \mathbf{\Sigma} = A^\sigma g_{\sigma\nu} \Sigma^\nu(\mathbf{A}, \mathbf{B}, \mathbf{C}) \tag{4}$$

$$= A^\sigma g_{\sigma\nu} g^{\nu\mu} \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma = 0, \tag{5}$$

and similarly for the others.

3. On the other hand, its inner product with the current 4-vector, Eq.(1),

$$\mathbf{S} \cdot \mathbf{\Sigma} = S^\sigma \underbrace{\mathbf{e}_{\sigma\nu} \cdot \mathbf{e}_\nu g^{\nu\mu}}_{\delta_\sigma{}^\mu} \Sigma_\mu(\mathbf{A}, \mathbf{B}, \mathbf{C})$$

yields

$$\mathbf{S} \cdot \mathbf{\Sigma}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = {}^*\mathbf{S}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \tag{6}$$

$$= \begin{pmatrix} \text{\# of particle world lines} \\ \text{passing through the volume} \\ \text{spanned by } \mathbf{A}, \mathbf{B}, \mathbf{C} \end{pmatrix}. \tag{7}$$

Thus the particle number ${}^*\mathbf{S}$ arises from the particle 4-current $\mathbf{S}$ and the "volume vector" $\mathbf{\Sigma}$ conjointly.

4. Fourth, the "volume vector"

$$\mathbf{\Sigma} = \mathbf{e}_\mu \underbrace{\epsilon^\mu{}_{\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}}_{\mathbf{\Sigma}^\mu} \tag{8}$$

[1] Indeed, one has

$$\begin{aligned} {}^{(2)}\mathbf{\Sigma}(\vec{A}, \vec{B}) &= \vec{e}_\ell \; \underbrace{{}^{(2)}\Sigma^\ell(\vec{A}, \vec{B})}_{g^{\ell i} \underbrace{{}^{(2)}\Sigma_i(\vec{A}, \vec{B})}_{\epsilon_{ijk} \frac{dx^j \wedge dx^k}{2!}(\vec{A}, \vec{B})}} \\ &= \vec{e}_\ell \; g^{\ell i} \epsilon_{ijk} A^j B^k \\ &= \vec{e}_\ell \; (\vec{A} \times \vec{B})^\ell \\ &\equiv \vec{A} \times \vec{B} \end{aligned}$$

is invariant under translations into any direction; in other words, its covariant differential vanishes:

$$d(\mathbf{e}_\mu \mathbf{\Sigma}^\mu) = 0$$

Leaving the spanning vectors in Eq.(6) unspecified, capitalize on this invariance to find that the exterior derivative of the particle density-flux 3-form is

$$\begin{aligned}
d\,{}^*\mathbf{S} &= d\left(\overbrace{\mathbf{e}_\mu S^\mu}^{\mathbf{S}} \cdot \overbrace{\mathbf{e}_\nu \Sigma^\nu}^{\mathbf{\Sigma}}\right) \\
&= d(\mathbf{e}_\mu S^\mu) \cdot \mathbf{e}_\nu \wedge \Sigma^\nu + S^\mu \mathbf{e}_\mu \cdot d\left(\mathbf{e}_\nu \Sigma^\nu\right) \\
&= \mathbf{e}_\mu S^\mu{}_{;\sigma} dx^\sigma \cdot \mathbf{e}_\nu \wedge \Sigma^\nu + zero \\
&= S^\mu{}_{;\sigma}\, \delta^\sigma{}_\mu \sqrt{-{}^4g}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= \frac{\partial\left(S^\mu \sqrt{-{}^4g}\right)}{\partial x^\mu} d^4x \\
&= S^\mu{}_{;\mu} \sqrt{-{}^4g}\, d^4x
\end{aligned} \tag{9}$$

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