

LECTURE 10

The Density-Flux 3form:

I) Algebraic properties

II) Four geometrical properties

In MTW grasp the ideas in

Box 4.4

Box 5.1

Fig. 5.1

Box 5.2

Geometric Properties of the Particle Density-Flux 3-form

$$*S = N U^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}$$

I) Its Multilinear (Algebraic) Properties

This $\binom{0}{3}$ tensor is a trilinear function

$$*S : V \times V \times V \rightarrow \mathbb{R} \quad (= \text{particle } \#)$$

$$(A, B, C) \mapsto *S(A, B, C) = N U^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} (A, B, C)$$

$\underbrace{\quad}_{S^\mu}$ $\underbrace{\quad}_{3!}$
 particle current 4-vector component $\Sigma_\mu(A, B, C)$
 "volume 1-form" component

$$\sum_{\mu} (A, B, C) \vec{E}_\mu(A, B, C) dx^\mu$$

The vectors (A, B, C) spanning this

"volume 1-form" also span the corresponding "volume vector,"

$$\Sigma(A, B, C) = e_\nu g^{\nu\mu} \underbrace{\epsilon_{\mu\alpha\beta\gamma}}_{\sqrt{-\det g} [\mu\alpha\beta\gamma]} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} (A, B, C)$$

$$\sqrt{-\det g} [\mu\alpha\beta\gamma]$$

4-d Levi-Civita
tensor component

$$\equiv e_\nu \Sigma^\nu(A, B, C)$$

This "volume vector" a vector-valued 3-form

$$\Sigma = e_\nu \Sigma^\nu \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}$$

is, as we shall see, a fundamental
building block for spacetime frame

-invariant integrals that mathematize
particle number, charge, momentum,
etc.

The "volume vector" Σ has a number
of key properties which geometrize

these physical properties.

Before highlighting ^{them} one needs to concretize the Levi-Civita tensor

components

$$\epsilon_{\mu\alpha\beta\gamma} = \sqrt{-g} [\mu\alpha\beta\gamma],$$

Here $[\mu\alpha\beta\gamma]$ is the antisymmetric permutation symbol whose value is 1, -1, or 0, and g is the determinant of the metric relative to a generic coord. basis whose relation to the Minkowski

basis is

$$\begin{aligned} g_{\mu\nu} &= \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \eta_{\alpha\beta} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \\ &= \Lambda^\alpha_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu \\ &= (\Lambda^T \eta \Lambda)_{\mu\nu} \end{aligned}$$

and

$$\det g = \det \Lambda^T \det \eta \det \Lambda$$

$$= -(\det \Lambda)^2$$

Consequently, the Jacobian determinant

$$\det \left(\frac{\partial \bar{x}^\beta}{\partial x^\alpha} \right) = \sqrt{-\det g} \equiv \sqrt{-g}$$

Comment:

The 4-d spacetime integrand:

$$\sqrt{-g} d^4x = \sqrt{-g} [\mu \alpha \beta \gamma] \underbrace{dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma}_{4!}$$

$$= d\bar{x}^0 \wedge d\bar{x}^1 \wedge d\bar{x}^2 \wedge d\bar{x}^3 \equiv d^4\bar{x}$$

is a frame invariant. Its value is

simply the Minkowski 4-d volume element in the LAB frame when one changes variables in 4-d multiple integrals,

$$\iiint \dots d^4x = \iiint \dots \frac{\partial(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)}{\partial(x^0, x^1, x^2, x^3)} dx^0 dx^1 dx^2 dx^3$$

$$= \iiint \dots \sqrt{-g} d^4x$$

II) Its Geometric (Inner Product) Properties

The particle density-flux 3-form

$$\begin{aligned}
 {}^*S(A, B, C) &= N u^\mu \epsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!} (A, B, C) \\
 &= S^\mu \Sigma_\mu (A, B, C) \\
 &= S^\mu g_{\mu\nu} \epsilon^{\nu\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!} (A, B, C) \\
 &= \underbrace{S^\mu e_\mu \cdot e_\nu}_{S \cdot} \Sigma^\nu (A, B, C) \\
 &= S \cdot \Sigma (A, B, C) \quad (10.1)
 \end{aligned}$$

is a multi-faceted concept which condenses the fundamental attributes of matter and of motion into the mathematical mapping *S .

Its two major facets are the particle 4-current

$$S = N u \equiv \underbrace{N u^\mu}_{S^\mu} e_\mu \quad (10.2)$$

and the "volume vector"

$$(10.3) \quad \Sigma = \mathcal{E}_\gamma \Sigma^\gamma = \mathcal{E}_\gamma \epsilon^\gamma_{\alpha\beta\delta} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\delta}{3!}.$$

It is a vector-valued 3-form.

Together they mathematize ^(the) different attributes

of moving matter. The focus of \mathcal{E}

is on the type of matter and its motion.

The focus of Σ is on the measurable

size of the spacetime domain occupied

by the matter in motion. To mathematize its

accommodations in a frame independent

way one must do so in terms of a

4-vector, the "volume vector" Σ , a

relativistically invariant concept.

At its core this "volume vector" has four geometrical properties.

1. First of all, $\Sigma(A, B, C)$, Eq.(10.1) on page 10.5, is in 4-d spacetime the Lorentzian version of the "area vector", i.e., the familiar cross product in 3-d Euclidean space,

$$\begin{aligned}
 {}^{(2)}\Sigma(\vec{A}, \vec{B}) &= e_2 {}^{(2)}\Sigma^2(\vec{A}, \vec{B}) \\
 &= e_2 g^{2i} \underbrace{\Sigma_i(\vec{A}, \vec{B})}_{\epsilon_{ijk} \frac{dx^j dx^k}{2!}(\vec{A}, \vec{B})} \\
 &= e_2 g^{2i} \underbrace{\epsilon_{ijk} A^j B^k}_{(\vec{A} \times \vec{B})^k} \\
 &= e_2 (\vec{A} \times \vec{B})^2 \\
 &= \vec{A} \times \vec{B}
 \end{aligned}$$

Comment:

The sequence of steps leading to $\vec{A} \times \vec{B}$ is a frame independent process.

Relative to rectangular (x, y, z) coordinates

for an area subtended by an

as-yet-unspecified pair of vectors, the

"area vector" is

$$(2) \vec{\Sigma} = \vec{i} dy \wedge dz + \vec{j} dz \wedge dx + \vec{k} dx \wedge dy.$$

2. Secondly, just as in ^(3-d) Euclidean space the "area vector", i.e. the cross product $\vec{A} \times \vec{B}$ is orthogonal to the two spanning vectors

$$\vec{A} \cdot \underbrace{\Sigma(\vec{A}, \vec{B})}_{\vec{A} \times \vec{B}} = \vec{B} \cdot \underbrace{\Sigma(\vec{A}, \vec{B})}_{\vec{A} \times \vec{B}} = 0,$$

one has (Lorentz) orthogonality for each of the spanning vectors.

$$\begin{aligned} A \cdot \Sigma(A, B, C) &= A^\sigma g_{\sigma\gamma} \Sigma^\gamma(A, B, C) \\ &= A^\sigma E_{\sigma\alpha\beta\gamma} A^\alpha B^\beta C^\beta = 0, \end{aligned}$$

and similarly for B and C.

3. On the other hand, the inner product with the particle 4-current, Eq. (10.2) on page 10.5 yields in light of Eq. (10.1) the result that

$$S \cdot \Sigma(A, B, C) = {}^*S(A, B, C)$$

= # of particle world line passing thru the volume spanned by A, B, and C,

4. Fourth, the "volume vector", Eq. (10.3) on page 10.6, is invariant under translation into any direction. In other words, its covariant differentiated

$$0 = d(e_\mu \Sigma^\mu) = d(e_\mu g^{m\nu} e_\nu \epsilon_{\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!})$$

vanishes,

The density-flux 3-form *S is the inner product of the particle 4-current, Eq. (10.2), on page 10.5 and that "volume vector":

$${}^*S = S^\sigma e_\sigma \cdot e_\mu \Sigma^\mu$$

By capitalizing on translation invariance of $e_\mu \Sigma^\mu$ one finds that the (covariant) differential of *S is

$$\begin{aligned} d({}^*S) &= d(e_\sigma S^\sigma \cdot e_\mu \Sigma^\mu) \\ &= d(e_\sigma S^\sigma) \cdot e_\mu \Sigma^\mu + e_\sigma S^\sigma \wedge d(e_\mu \Sigma^\mu) \\ &= e_\sigma S^\sigma{}_{; \rho} dx^\rho \cdot e_\mu \Sigma^\mu + \text{zero} \\ &= S^\sigma{}_{; \rho} S^\rho \sqrt{-g} dx^0 dx^1 dx^2 dx^3 \end{aligned}$$

where

$$S^\sigma{}_{; \rho} = \frac{\partial S^\sigma}{\partial x^\rho} + S^\nu \Gamma^\sigma{}_{\nu\rho}$$

are the components of the covariant derivative

of the particle 4-current S , it follows that

$$\begin{aligned}
 d^{(4)}\mathcal{V} &= S^{\sigma}_{;\sigma} \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
 &= \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} S^{\sigma})}{\partial x^{\sigma}} \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
 &= \left(\begin{array}{l} \# \text{ of particles} \\ \text{created in the} \\ \text{invariant} \\ \text{spacetime} \\ \text{4-volume } \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{array} \right)
 \end{aligned}$$

Looking ahead, one finds that whenever particles are neither created nor destroyed, then such a state of affairs is mathematized by the

statement

$$\boxed{\frac{\partial(\sqrt{-g} S^{\sigma})}{\partial x^{\sigma}} = 0}$$