

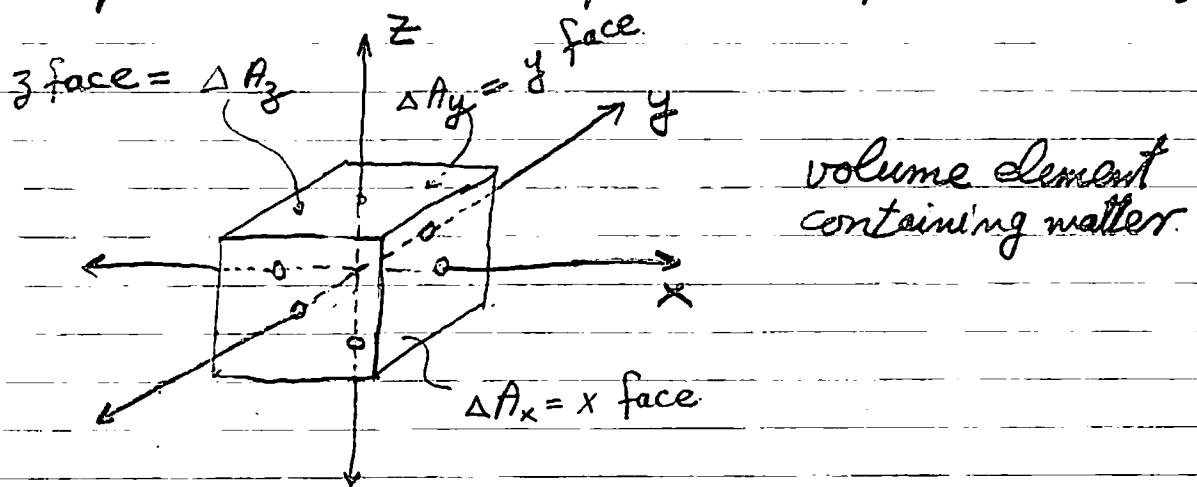
LECTURE 11 (Appendix)

Why is the stress tensor symmetric?

The spatial stress components form a symmetric matrix

$$T^{zj} = T^{ji}$$

One arrives at this conclusion with the help of Newton's equation applied to the rotational motion of a small cube of matter of volume L^3 .



The mass in such a cube is $T^{00} L^3$. The moment of inertia of this cube is $\sim T^{00} L^5$.

Newton's equation of motion applied to the cube's rotation around the z-axis is

$$\dot{\Omega}(T^{00}L^5) = (\text{torque})^3 = (\vec{r} \times \vec{\Delta F})^3 = [\sum r_z \Delta F_y] - [\sum r_y \Delta F_z]$$

$$= \left[\underbrace{\frac{L}{2}}_{\text{"on" or "applied to" }} \underbrace{T^{yx}(-)L^2}_{\substack{\text{lever arm} \\ \text{to (+)x face}}} + \underbrace{(-)\frac{L}{2}}_{\substack{\text{lever arm} \\ \text{to (-)x face}}} \underbrace{T^{yx}L^2}_{\substack{\text{y-force exerted} \\ \text{by surrounding} \\ \text{on (+)x face}}} \right]$$

"or" or "applied to"

$$- \left[\underbrace{\frac{L}{2}}_{\substack{\text{lever arm} \\ \text{to (+)y face}}} \underbrace{T^{xy}(-)L^2}_{\substack{\text{x-force by surrounding} \\ \text{on (+)y face}}} + \underbrace{(-)\frac{L}{2}}_{\substack{\text{lever arm} \\ \text{to (-)y face}}} \underbrace{T^{xy}L^2}_{\substack{\text{x-force by surrounding} \\ \text{on (-)y face}}} \right]$$

lever arm y-force exerted by surrounding on (+)x face.

lever arm x-force by surrounding on (+)y face.

Comments:

a) Note that at $x = \frac{L}{2}$

$$\Delta F^y = T^{yx} \Delta A_x = T^{yx} L^2$$

is a y force exerted by the x -face on the outside medium ($\frac{L}{2} < x$). This is because this y force expresses a flow of y -momentum into the $+x$ direction at the $+x$ face.

b) Equivalently

$$(-) T^{yx} L^2.$$

is a y force exerted on the x -face by the outside medium. This expresses a flow of y -momentum into the $-x$ direction at the $+x$ face.

In other words, it is the direction of the flow of momentum that that gets reversed when one reverses the origin and the destination of the application of a force.

c) At the $-x$ face $T^{yx} L^2$ also expresses a flow of y momentum into the x direction. But here it represents a y force exerted on the $-x$ face by the outside medium ($x = -\frac{L}{2}$); in other words, the momentum flow T^{yx} is from the outside to the inside of the cube across the $-x$ face.

d) As an aside, we note that the y -force on the $+x$ face can be represented in terms of the vector valued 3-form *T by

${}^*T(A', B, C)$ where

$$A' : \begin{smallmatrix} t & x & y & z \\ 1 & 0 & 0 & 0 \end{smallmatrix}$$

$$B : (0, 0, L, 0)$$

$$C : (0, 0, 0, L)$$

as follows

$$\begin{aligned} (-)T^{yx} \Delta A_x &= (-)T^{yx} L^2 = (-)T^{yx} (-)E_{xyyz} 1 \cdot L \cdot L \quad [E_{xyyz}] \\ &= T^{yx} E_{xyyz} \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma (A', B, C) \end{aligned}$$

Flow of y momentum across ΔA_x into negative direction

(into the cube in the picture on P10)

e) In fact, more generally we note that
 the spatial components of the force $\vec{\Delta F}$ together
 with the energy rate ("power")

$$\sum T^{0i} \Delta A_i = T^{0x} \Delta A_x + T^{0y} \Delta A_y + T^{0z} \Delta A_z$$

make up the components of the 4-momentum

$$e, T^{0i} \Delta A_i = \sum_A P_A N_A U_A^\nu E_{\nu 0ij} 1 B^i C^j = \leftrightarrow T(A', B, C)$$

crossing the element of area $\vec{\Delta A} = \vec{B} \times \vec{C}$

during the time of the vector $A' : (1, 0, 0, 0)$

By contrast the 4-momentum in the future

directed 3-volume $L^3 = \vec{A} \cdot \vec{B} \times \vec{C}$ is

$$T^{00} L^3 = \sum_A P_A N_A U_A^\nu E_{\nu 0ij} 1^i B^j C^k = \leftrightarrow T(A, B, C)$$

where $A : (A^0, \vec{A})$

$B : (B^0, \vec{B})$

$C : (C^0, \vec{C})$

are spacelike.

Back to Newton's eq'n on II, A1:

II, A5

After simplifying Newton's equations for cube's rotation around the z -axis (P 10) one obtain

$$(\dot{\bar{\Omega}})^3 = \frac{T^{xy} - T^{yx}}{T^{xx}} \frac{L^3}{L^5}$$

We see that if it were true that that $T^{xy} \neq T^{yx}$, then

$$(\dot{\bar{\Omega}})^3 \rightarrow \infty \text{ as } L \rightarrow 0.$$

In other words, the cube of matter would be spinning with arbitrarily large angular velocities if we would consider the cube to be of sufficiently small volume L^3 . The fact that matter does not behave that way demands that

$$T^{xy} = T^{yx}$$

or more generally

$$T^{ij} = T^{ji}$$