

LECTURE 12

- I) Stress tensor as the area-force relation

Read MTW: Chapter 5, p131; § 5.3, p138

- II) a) The "area vector" as a vectorial 2-form in 3-d.

- b) Surface integral in 3-d; How to construct it relative to generic curvilinear coordinates.

- c) Example for a spherical surface

All matter is composed of particles.

Their averaged motion and/or interaction across a small area manifests itself as a force on this area. Moreover, for small areas (but still large enough to preserve the validity of the averaging process) the relation between the size of the area and the force across it is a linear one.

I) Force-area Relation

Consider a volume element with its bounding surface areas.

Each of them has a (spatial) normal, a vector, and also has a vectorial force acting on it. This force vector acting on the surface element characterized by its normal vector is a type of stress. The problem is to mathematize this stress.

This mathematization process is executed as follows:

A) ELEMENTS of AREA

12.3

Consider a laboratory coordinate frame coordinatized by (x, y, z) and an element of (triangular) area with vertices at $x = a_1, y = a_2, z = a_3$.

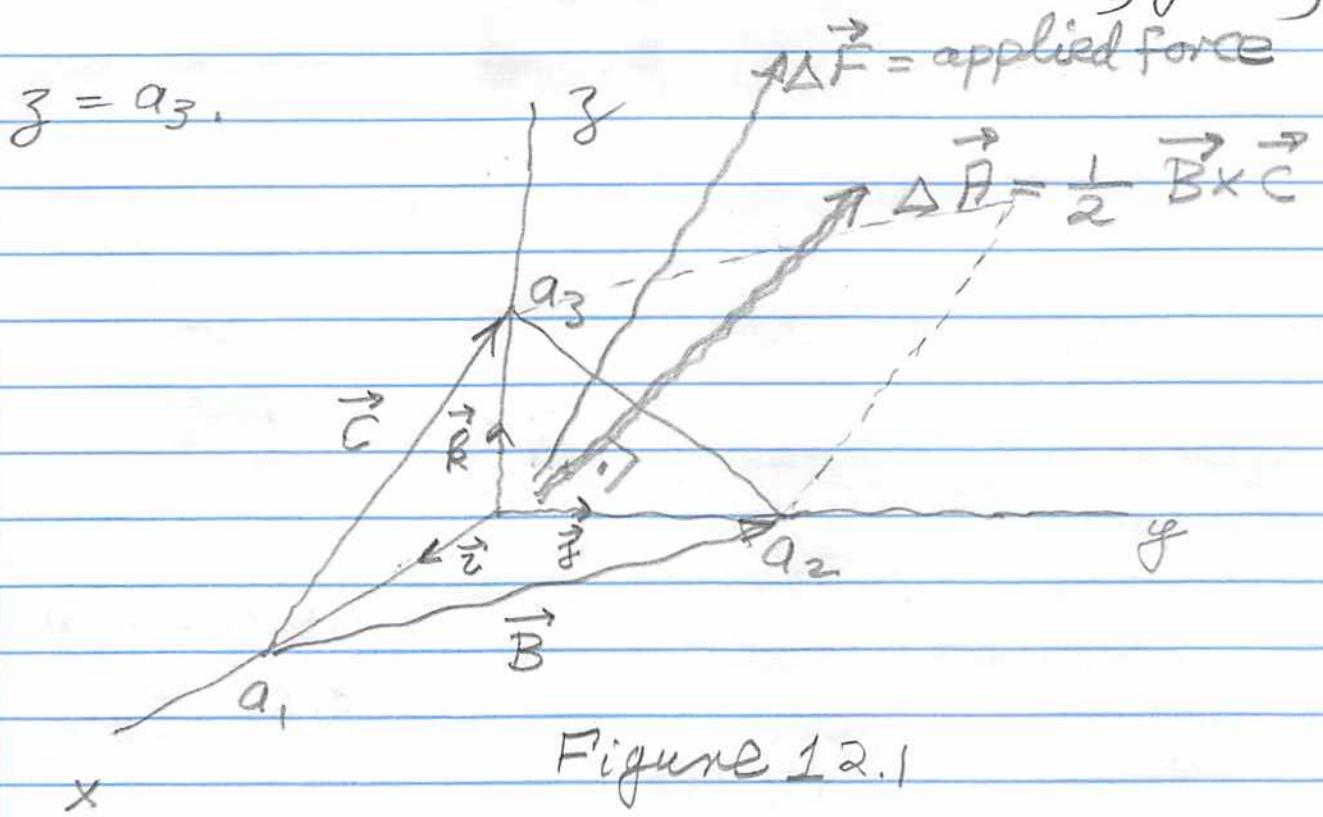


Figure 12.1

This area is subtended by the vectors

$$\vec{B} = \vec{x}a_2 - \vec{z}a_1$$

$$\vec{C} = \vec{z}a_3 - \vec{x}a_1$$

The vector normal to the area

is

$$\Delta \vec{A} = \frac{1}{2} \vec{B} \times \vec{C}$$

$$= \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a_1 & a_2 & 0 \\ -a_1 & 0 & a_3 \end{vmatrix}$$

$$= \frac{1}{2} [\vec{i} a_2 a_3 + \vec{j} a_3 a_1 + \vec{k} a_1 a_2]$$

$$= i dx^2 \wedge dx^3 (B, C) + j dx^3 \wedge dx^1 (B, C) + k dx^1 \wedge dx^2 (B, C)$$

Thus one has decomposed the normal

vector

$$\Delta \vec{A} = \vec{i} \Delta A_x + \vec{j} \Delta A_y + \vec{k} \Delta A_z$$

$$= \vec{e}_1 \Delta A_1 + \vec{e}_2 \Delta A_2 + \vec{e}_3 \Delta A_3$$

into its components

$$\Delta A_x = \frac{1}{2} a_2 a_3 = \frac{1}{2} \epsilon_{ijk} \frac{dx^i \wedge dx^k}{2!} (B, C)$$

$$\Delta A_y = \frac{1}{2} a_3 a_1 = \frac{1}{2} \epsilon_{2jk} \frac{dx^j \wedge dx^k}{2!} (B, C)$$

$$\Delta A_z = \frac{1}{2} a_1 a_2 = \frac{1}{2} \epsilon_{3jk} \frac{dx^j \wedge dx^k}{2!} (B, C)$$

relative to the lab basis ($\vec{i}, \vec{j}, \vec{k}$).

Here $\epsilon_{ijk} \frac{dx^i \wedge dx^k}{2!}$ is the i^{th} component of the "area vector".

These lab components are the projections of $\Delta \vec{F}$ onto the respective coordinate planes

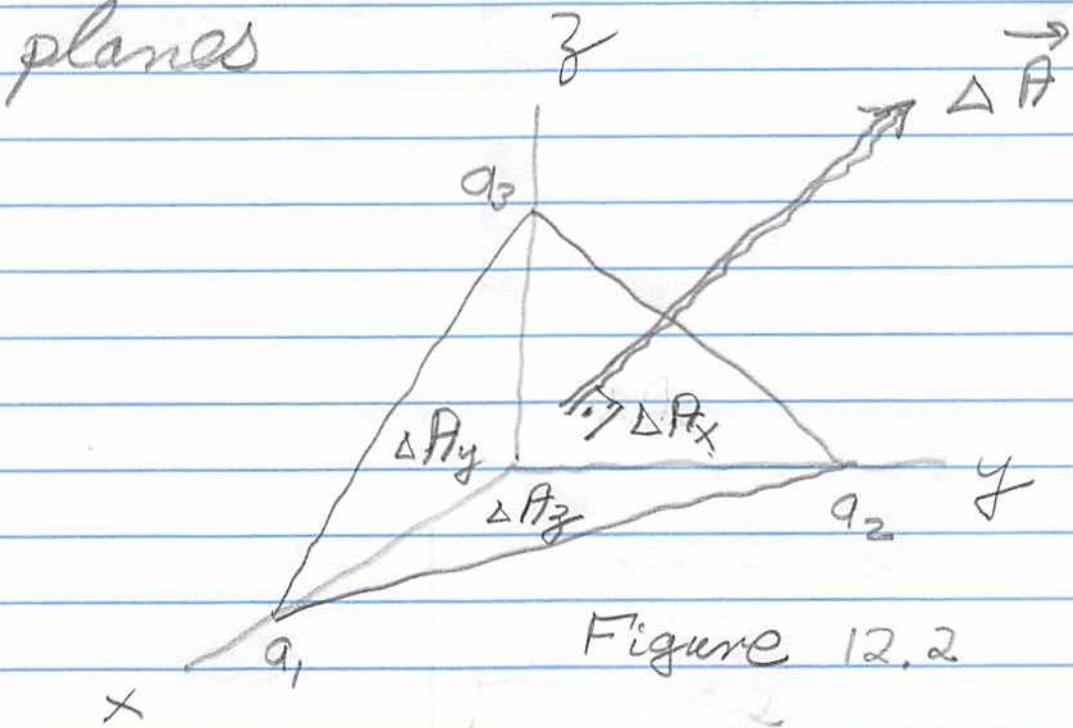


Figure 12.2

B.) STRESS

We now introduce a new concept, namely, the relationship between $\Delta \vec{F}$ and $\Delta \vec{A}$. The fact that the force field \vec{F} is distributed uniformly over the planar neighborhood which contains $\Delta \vec{A}$

implies that doubling the size of $\vec{\Delta A}$ doubles the size of $\vec{\Delta F}$. In other words, $\vec{\Delta F}$ is a linear function of $\vec{\Delta A}$,
 (i.e., a linear causal relationship between $\vec{\Delta A}$ and $\vec{\Delta F}$)
 This linear function is the stress to which the matter in the volume element is subjected

$$\vec{\Delta F} = \vec{i} \Delta F_x + \vec{j} \Delta F_y + \vec{k} \Delta F_z = \text{"stress"}(\vec{\Delta A}),$$

where "stress" is mathematized by the following equations

$$\Delta F_x = T^{xx} \Delta A_x + T^{xy} \Delta A_y + T^{xz} \Delta A_z = T^{yz} \Delta A_2$$

$$\Delta F_y = T^{yx} \Delta A_x + T^{yy} \Delta A_y + T^{yz} \Delta A_z = T^{xz} \frac{1}{2} \sum_{j=1}^2 E_{ijk} dx^j dx^k (B, c)$$

$$\Delta F_z = T^{zx} \Delta A_x + T^{zy} \Delta A_y + T^{zz} \Delta A_z = T^{xy} \frac{1}{2} \sum_{j=1}^2 E_{ijk} dx^j dx^k (B, c)$$

the above-mentioned

They comprise 3 linear causal relation-

$$\Delta F_E = T^{xx} \int_{B_1} d\mathbf{x}^k dx^1 dx^2 (B_1 C) \frac{1}{2}$$

$$*T = e_E T^{xx} \int_{B_1} d\mathbf{x}^k dx^1 dx^2$$

12.7

Each of the stress components T^{xx}, T^{xy} , etc. is measurable. They characterize the stress to which the matter is subjected in the neighborhood of the origin in Figures 12.1 and 12.3 on pages 2 and 4. For a given volume element of matter these components form a square array.

$$\text{"stress"} = \begin{bmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{bmatrix}$$

where each of the diagonal components is a pressure,

$$T^{xx} = \frac{(\text{Force into } x\text{-direction})}{(\text{unit area pointing into } x\text{-direction})} = \text{pressure into the } x\text{-direction}$$

while each of the off-diagonal elements is a shear stress,

$$T^{xy} = \frac{(\text{Force into } x\text{-direction})}{(\text{unit area pointing into } y\text{-direction})} = \text{shear stress}$$

In general

$$T^{ii} = \text{pressure (no sum)}$$

$$T^{ij} = \text{shear stress } (i \neq j)$$

II) "AREA VECTOR" AS A VECTORIAL 2-FORM

in 3-d SPACE.

The vector \perp to the area spanned by

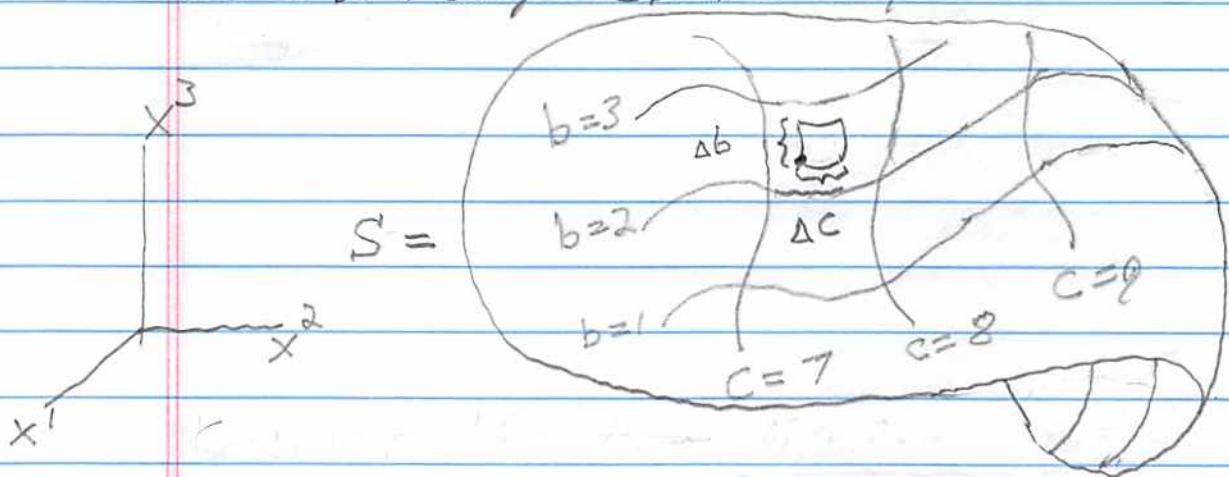
\vec{B} and \vec{C} relative any chosen basis $\{\vec{e}_i\}$ is

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \vec{B} \times \vec{C} = \vec{e}_k \epsilon_{mn}^R \frac{dx^m dx^n}{2!} (\vec{B}, \vec{C})$$

$$= \vec{e}_k \sum_R (\vec{B}, \vec{C})$$

$$g^{kl} \epsilon_{lmn} = g^{lk} \sqrt{\det g} [P_{mn}]$$

Introduce arbitrary coordinates b and c on a 2-d surface S in 3-d space



$$\vec{B} = \vec{e}_m B^m = \vec{e}_m \frac{\partial x^m}{\partial b} \Delta b \text{ and } \vec{C} = \vec{e}_n C^n = \vec{e}_n \frac{\partial x^n}{\partial c} \Delta c$$

On this surface consider the element
of area bounded by coordinate
lines

$$b_0 < b < b_0 + \Delta b$$

$$c_0 < c < c_0 + \Delta c$$

Its edges are the area's spanning
vectors

$$\vec{B} = \vec{e}_m B^m = \vec{e}_m \frac{\partial x^m}{\partial b} \Delta b$$

and

$$\vec{C} = \vec{e}_n C^n = \vec{e}_n \frac{\partial x^n}{\partial c} \Delta c,$$

Thus the "area vector" $\vec{e}_k \sum_m^k (\vec{B}, \vec{C})$ has

components

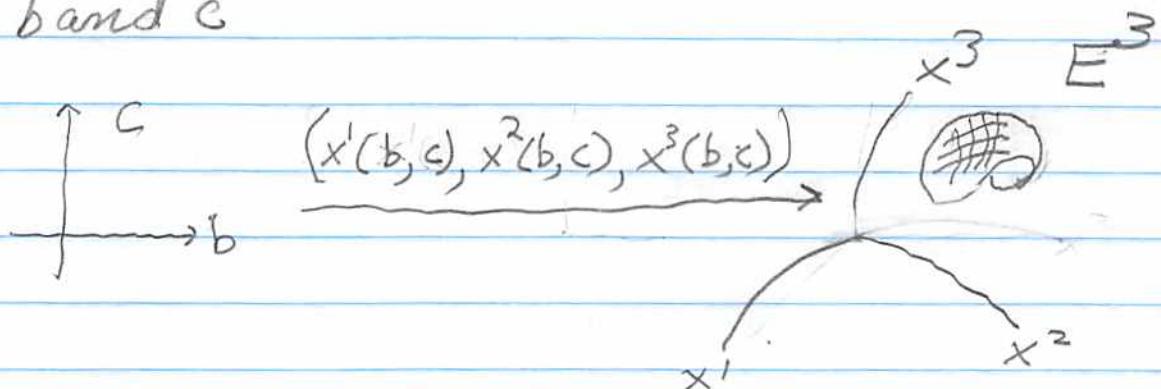
$$\sum_m^k (\vec{B}, \vec{C}) = \epsilon_{mn}^k \frac{\partial x^m}{\partial b} \frac{\partial x^n}{\partial c} \Delta b \Delta c \equiv d^2 \sum_m^k$$

and the corresponding "area 1-form"

$d^2 \sum_m^k (\vec{B}, \vec{C})$ has components

$$\underbrace{\sum_m^k (\vec{B}, \vec{C})}_{\epsilon_{lmn}} = \epsilon_{lmn} \frac{\partial x^l}{\partial b} \frac{\partial x^m}{\partial c} \Delta b \Delta c \equiv d^2 \sum_m^k$$

Application setting up a surface integral over the surface S , which is parametrized by b and c



Let $\overleftrightarrow{T} = \vec{e}_i T^{ij} \vec{e}_j$ a tensor field in E^3 .

The integral of \overleftrightarrow{T} over S , e.g. the total force on S due to the given stress field, has components

$$\begin{aligned} F^i &= \iint_S T^{i\bar{j}} d^2 \Sigma_{\bar{j}} \\ &= \iint_S T^{i\bar{j}} \epsilon_{\bar{j}mn} \frac{\partial x^m}{\partial b} \frac{\partial x^n}{\partial c} db dc \end{aligned}$$

$$\begin{aligned} (12.1) \quad &= \iint \left\{ T^{i1} \epsilon_{123} \left(\frac{\partial x^2 \partial x^3}{\partial b \partial c} - \frac{\partial x^3 \partial x^2}{\partial b \partial c} \right) \right. \\ &\quad + T^{i2} \epsilon_{231} \frac{\partial x^3 \partial x^1}{\partial b \partial c} - \frac{\partial x^1 \partial x^3}{\partial b \partial c} \\ &\quad \left. + T^{i3} \epsilon_{312} \frac{\partial x^1 \partial x^2}{\partial b \partial c} - \frac{\partial x^2 \partial x^1}{\partial b \partial c} \right\} db dc \end{aligned}$$

$$= \iint_S T^{ij} \sum_{m,n} \frac{\partial(x^m, x^n)}{\partial(a, b)} db da$$

where

$$\frac{\partial(x^m, x^n)}{\partial(a, b)} : \begin{cases} m = 1, 2, 3 \\ n = 1, 2, 3 \end{cases}$$

are the three 2×2 Jacobian sub determinants of
the rectangular matrix.

$$\begin{bmatrix} \frac{\partial x^1}{\partial b} & \frac{\partial x^2}{\partial b} & \frac{\partial x^3}{\partial b} \\ \frac{\partial x^1}{\partial c} & \frac{\partial x^2}{\partial c} & \frac{\partial x^3}{\partial c} \end{bmatrix}$$

They are exhibited in Eq.(12.1) on
page 12,11

Example: Integral over a spherical

surface with radius R



$$(x^1, x^2, x^3) = (x, y, z); \quad (b, c) = (\theta, \varphi)$$

$$x = R \sin \theta \cos \varphi$$

$$y = R \sin \theta \sin \varphi; \quad \det g = 1; \quad \epsilon_{xyz} = 1$$

$$z = R \cos \theta$$

$$\begin{bmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} R \cos \theta \cos \varphi & R \cos \theta \sin \varphi & -R \sin \theta \\ -R \sin \theta \sin \varphi & R \sin \theta \cos \varphi & 0 \end{bmatrix}$$

The Jacobian sub determinants are

$$\frac{d^2 \Sigma_x}{d\theta d\varphi} = \frac{\partial(y, z)}{\partial(\theta, \varphi)} = \frac{\partial y}{\partial \theta} \frac{\partial z}{\partial \varphi} - \frac{\partial z}{\partial \theta} \frac{\partial y}{\partial \varphi} = R^2 \sin^2 \theta \cos \varphi$$

$$\frac{d^2 \Sigma_y}{d\theta d\varphi} = \frac{\partial(z, x)}{\partial(\theta, \varphi)} = \frac{\partial z}{\partial \theta} \frac{\partial x}{\partial \varphi} - \frac{\partial x}{\partial \theta} \frac{\partial z}{\partial \varphi} = R^2 \sin^2 \theta \sin \varphi$$

$$\frac{d^2 \Sigma_z}{d\theta d\varphi} = \frac{\partial(x, y)}{\partial(\theta, \varphi)} = \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \varphi} - \frac{\partial y}{\partial \theta} \frac{\partial x}{\partial \varphi} = R^2 \cos \theta \sin \varphi$$

12.14

The "area vector" is

$$\vec{e}_z \sum^2 (\vec{B}, \vec{C}) = \vec{e}_z d^2 \sum^2 = \vec{e}_z \epsilon_{mn}^2 B^m C^n$$

$$= \vec{i} R^2 \sin^2 \theta \cos \phi d\theta d\phi + \vec{j} R^2 \sin^2 \theta \sin \phi d\theta d\phi$$

$$+ \vec{k} R^2 \sin \theta \cos \theta d\theta d\phi$$

Introducing the radial unit vector

$$\vec{e}_r = \vec{i} \sin \theta \cos \phi + \vec{j} \sin \theta \sin \phi + \vec{k} \cos \theta,$$

one finds that this infinitesimal

"area vector" is simply

$$\boxed{\vec{e}_r d^2 \sum^2 = \vec{e}_r R^2 \sin \theta d\theta d\phi.}$$

This is the familiar element of area on
a sphere of radius R^2 .

12/15

The three components integrated over the surface are:

$$F^x = \iint (T^{xx} d\Sigma_x + T^{xy} d\Sigma_y + T^{xz} d\Sigma_z)$$

$$= R^2 \iint \{ T^{xx} \sin^2 \theta \cos \phi + T^{xy} \sin^2 \theta \sin \phi + T^{xz} \cos \theta \sin \theta \} d\theta d\phi$$

$$F^y = \iint (T^{yx} d\Sigma_x + T^{yy} d\Sigma_y + T^{yz} d\Sigma_z)$$

$$= R^2 \iint T^{yx} \sin^2 \theta \cos \phi + T^{yy} (-\sin^2 \theta \sin \phi) + T^{yz} \cos \theta \sin \theta \} d\theta d\phi$$

$$F^z = \iint (T^{zx} d\Sigma_x + T^{zy} d\Sigma_y + T^{zz} d\Sigma_z)$$

$$= R^2 \iint \{ T^{zx} \sin^2 \theta \cos \phi + T^{zy} (-\sin^2 \theta \sin \phi) + T^{zz} \cos \theta \sin \theta \} d\theta d\phi$$