

LECTURE 13

I) A) Momentenergy for a mixture of particles;

B) *S vs S

II) Conservation of momentenergy,

In MTW Read §5.8, 5.9, 5.10

I MOMENERGY FOR A MIXTURE OF PARTICLES

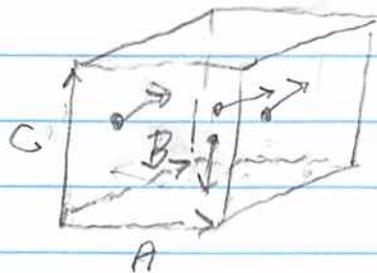
A) Q: What is the momenergy carried by an aggregate of different particles moving with different 4-velocities?

A: Consider a 3-d spacetime element of volume spanned by the triad of 4-d vector (A, B, C) and populated by different particles having their own rest masses m_1, m_2, \dots

4-velocities u_1, u_2

momenta p_1, p_2

invariant densities N_1, N_2, \dots



The total amount of momentum due to these particles in the (A, B, C) spanned volume is

$$\begin{aligned}
 * T(A, B, C) &= \sum_a p_a^* \bar{S}_a(ABC) = \sum p_a N_a u_a^\nu \epsilon_{\nu\alpha\beta\gamma}(ABC) \\
 &= e_\mu \left(\underbrace{\sum p_a^\mu N_a u_a^\nu}_{T^{\mu\nu}} \right) \underbrace{\epsilon_{\nu\alpha\beta\gamma}(ABC)}_{d^3\Sigma_\nu(A, B, C)}
 \end{aligned}$$

Here

$$1.) \quad T^{\mu\nu} = \sum_a p_a^\mu N_a u_a^\nu = \sum_a m_a N_a u_a^\mu u_a^\nu$$

are the components of the total momentum

Tensor

$$T = e_\mu \sum_a m_a N_a u_a^\mu u_a^\nu e_\nu$$

$$= e_\mu \otimes e_\nu T^{\mu\nu}$$

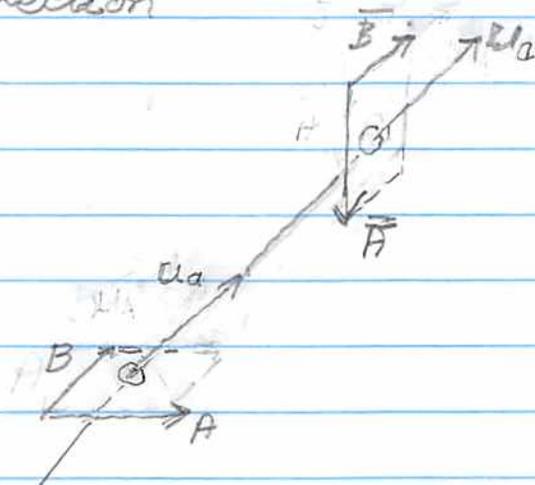
and

$$2.) \quad N_a u_a^\nu d^3\Sigma_\nu(A, B, C) = N_a u_a^\nu \epsilon_{\nu\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!}(A, B, C)$$

is the number of type "a" particle

which pierce the positively oriented

3-volume (A, B, C) in positively (i.e. in the same direction



B) Comment about "orientation"

The difference between \int

$$*\int = N u^\nu \epsilon_{\nu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{\partial x^\nu}$$

and $\int = N u^\nu \epsilon_\nu$

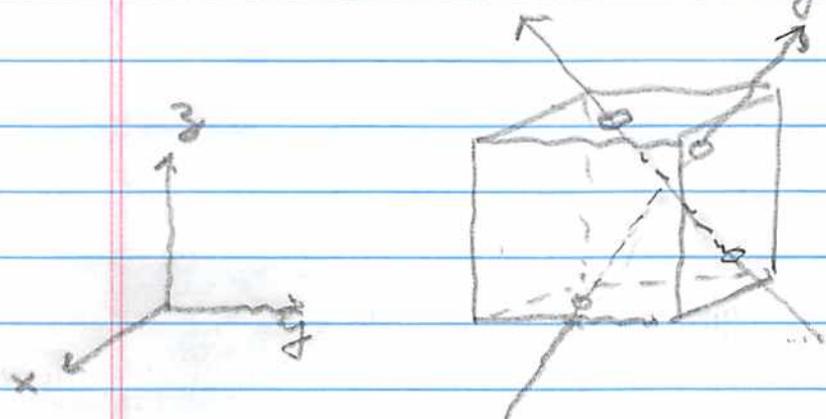
is that the concept $*\int$ includes a specification of orientation by means of the orientation

tensor (a.k.a. Levi-Civita) tensor:

$$\epsilon = \epsilon_{\nu\alpha\beta\gamma} dx^\nu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

while \int does not.

II.) Our interest lies in exhibiting a mathematical expression for the amount of momentum created in the volume $\Delta x \Delta y \Delta z$ during time Δt



We shall determine

(amount of momentum
created in a specified
4-d volume $\Delta t \Delta x \Delta y \Delta z$
of spacetime) $\equiv Q$

$$Q = \left(\text{momentum energy created in } \Delta x \Delta y \Delta z \right)$$

$$= \left(\text{change in m.e. in } \Delta x \Delta y \Delta z \text{ during time } \Delta t \right) + \left(\text{outflow of m.e. through the sides of } \Delta x \Delta y \Delta z \text{ during } \Delta t \right)$$

$$= \left(\text{m.e. in } \Delta x \Delta y \Delta z \text{ at the end, } t + \Delta t, \text{ of time interval } \Delta t \right) - \left(\text{m.e. in } \Delta x \Delta y \Delta z \text{ at the beginning, } t, \text{ of time interval } \Delta t \right)$$

$$\left(\text{flow of m.e. out of right hand face of } \Delta x \Delta y \Delta z \text{ during } \Delta t \right) + \left(\text{flow of m.e. out of left hand face of } \Delta x \Delta y \Delta z \text{ during } \Delta t \right)$$

$$+ \left(\text{flow of m.e. out of front face of } \Delta x \Delta y \Delta z \text{ during } \Delta t \right) + \left(\text{flow of m.e. out of back face of } \Delta x \Delta y \Delta z \text{ during } \Delta t \right)$$

$$+ \left(\begin{array}{l} \text{flow of m.e. out} \\ \text{of top} \\ \text{face of } \Delta x \Delta y \Delta z \\ \text{during } \Delta t \end{array} \right) + \left(\begin{array}{l} \text{flow of m.e. out} \\ \text{of bottom} \\ \text{face of } \Delta x \Delta y \Delta z \\ \text{during } \Delta t \end{array} \right)$$

Comment:

The total momentum flow across the 6 faces of $\Delta x \Delta y \Delta z$ can be positive or negative.

Taking note of the fact that in nature

$Q = 0$, one sees

e.g.

(change in m.e.) $< 0 \iff$ net out flow > 0
(in $\Delta x \Delta y \Delta z$)

" $> 0 \iff$ net out flow < 0
"inflow.

The mathematical expression for the four contributing pairs to Q is

$$Q = e_{\mu} T^{\mu 0} \Big|_{t+\Delta t}^{\Delta x \Delta y \Delta z} - e_{\mu} T^{\mu 0} \Big|_t^{\Delta x \Delta y \Delta z}$$

$$+ e_{\mu} T^{\mu x} \Big|_{x+\Delta x}^{\Delta y \Delta z \Delta t} + e_{\mu} T^{\mu x} \Big|_x^{(-) \Delta y \Delta z \Delta t}$$

$$+ e_{\mu} T^{\mu y} \Big|_{y+\Delta y}^{\Delta z \Delta x \Delta t} + e_{\mu} T^{\mu y} \Big|_y^{(-) \Delta z \Delta x \Delta t}$$

$$+ e_{\mu} T^{\mu z} \Big|_{z+\Delta z}^{\Delta x \Delta y \Delta t} + e_{\mu} T^{\mu z} \Big|_z^{(-) \Delta x \Delta y \Delta t}$$

$$Q = \frac{\partial}{\partial t} (e_{\mu} T^{\mu 0}) \Delta t \Delta x \Delta y \Delta z$$

$$+ \frac{\partial}{\partial x} (e_{\mu} T^{\mu x}) \Delta x \Delta y \Delta z \Delta t$$

$$+ \frac{\partial}{\partial y} (e_{\mu} T^{\mu y}) \Delta y \Delta z \Delta x \Delta t$$

$$+ \frac{\partial}{\partial z} (e_{\mu} T^{\mu z}) \Delta z \Delta x \Delta y \Delta t = \frac{\partial}{\partial x^{\nu}} (e_{\mu} T^{\mu \nu}) \Delta t \Delta x \Delta y \Delta z$$

rectilinear

(t, x, y, z) -induced

-13, 8-

Relative to the parallel basis one has

$$\nabla_{e_\nu} e_\mu = \frac{\partial e_\mu}{\partial x^\nu} = 0$$

Thus the created momentum / 4 volume
is

$$\frac{Q}{\Delta t \Delta x \Delta y \Delta z} = e_\mu T^{\mu\nu}$$

However, in nature momentum is
neither created ($Q > 0$) nor destroyed

($Q < 0$): $\boxed{Q = 0}$

The mathematization of this fact
consists of the statement

$$\boxed{\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0}$$