

LECTURE 14

I ORIENTATION

II Conservation of Momentum

mathematized by exterior differentiation

I.) ORIENTATION

In addition to an inner product, a vector space accommodates other geometrical structures besides an inner product. "Orientation" is another geometrical ornament with which a given vector space may be decorated.

The geometrical concept "orientation" is mathematized algebraically by means of the Levi-Civita tensor on the vector space. This is done as follows:

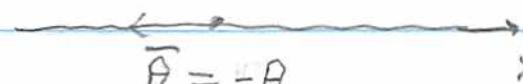
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1. Orientation on $V = \mathbb{R}^1$

Let $\sum = e, dx' \in V^*$
Then

a)  $A = A' \frac{d}{dx'} \in V$

so that $\sum(A) = e, dx'(A) = e, A'$
but

b)  $\bar{A} = -A \quad \bar{x}' \quad \bar{A} = -A' \frac{d}{dx'} \in V$

so that

$$\sum(\bar{A}) = e, dx'(\bar{A}) = -e, A'$$

Thus

$\{A\}$ and $\{\bar{A}\}$ have opposite

orientation on $V = \mathbb{R}^1$.

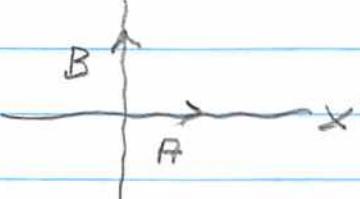
2. Orientation on $V = \mathbb{R}^2$

Let

$$\sum = \epsilon_{ij} \frac{dx^i dx^j}{2!} \in V^* \wedge V^* \subset \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

where $\epsilon_{ij} = \begin{cases} \pm \epsilon_{12} \\ 0 \end{cases}$

Then

a)  $A = A^i \vec{e}_i$
 $B = B^j \vec{e}_j$

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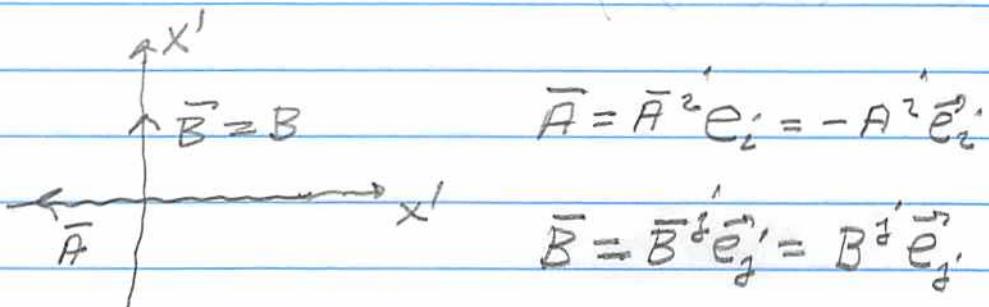
so that

$$\sum_{\text{m}}(A, B) = \det \begin{vmatrix} A^1 A^2 \\ B^1 B^2 \end{vmatrix} = \epsilon_{i_1 i_2 j_1 j_2} A^{i_1} B^{j_1} > 0$$

= (area spanned by (A, B))

but

b)



$$\bar{A} = \bar{A}^i e_i = -A^i \bar{e}_i$$

$$\bar{B} = \bar{B}^j \bar{e}_j = B^j \bar{e}_j$$

so that

$$\sum_{\text{m}}(\bar{A}, \bar{B}) = \det \begin{vmatrix} \bar{A}^1 \bar{A}^2 \\ \bar{B}^1 \bar{B}^2 \end{vmatrix}$$

$$= \epsilon_{i_1 i_2 j_1 j_2} \bar{A}^{i_1} \bar{B}^{j_1} = -\epsilon_{i_1 i_2 j_1 j_2} A^{i_1} B^{j_1} < 0$$

= (area spanned by (\bar{A}, \bar{B}))

Thus

$\{A, B\}$ and $\{\bar{A}, \bar{B}\}$ have opposite

orientation.

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3, Orientation in \mathbb{R}^3

Let

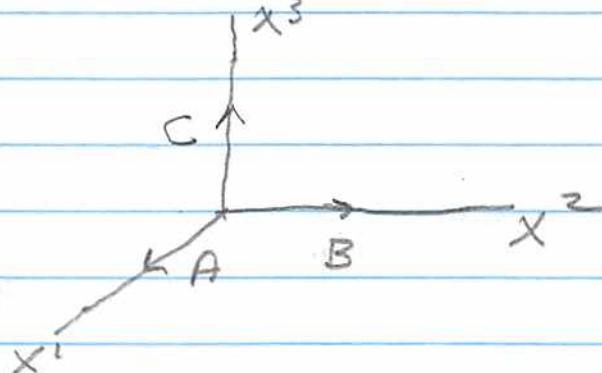
$$\sum = \epsilon_{ijk} \frac{dx^i \wedge dx^j \wedge dx^k}{3!} \in V \wedge V \wedge V \subset \binom{V}{3}$$

where

$$\epsilon_{ijk} = \begin{cases} \pm \epsilon_{123} \\ 0 \end{cases}$$

Then

a)



$$A = A^i \vec{e}_i$$

$$B = B^j \vec{e}_j$$

$$C = C^k \vec{e}_k$$

so that

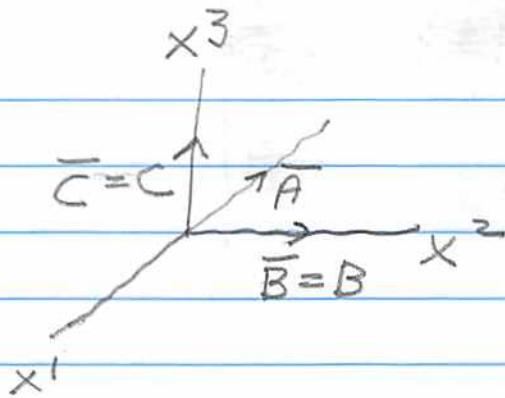
$$\sum_{ijk} (A, B, C) = \det \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} = \epsilon_{ijk} A^i B^j C^k > 0$$

\approx (volume spanned)
by (A, B, C)

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but

b)



$$\bar{A} = \bar{A}^i \vec{e}_i = -A^i \vec{e}_i$$

$$\bar{B} = \bar{B}^j \vec{e}_j = B^j \vec{e}_j$$

$$\bar{C} = \bar{C}^k \vec{e}_k = C^k \vec{e}_k$$

so that

$$\sum (\bar{A}, \bar{B}, \bar{C}) = \det \begin{vmatrix} \bar{A}^1 & \bar{A}^2 & \bar{A}^3 \\ \bar{B}^1 & \bar{B}^2 & \bar{B}^3 \\ \bar{C}^1 & \bar{C}^2 & \bar{C}^3 \end{vmatrix} = -\epsilon_{ijk} A^i B^j C^k < 0$$

= (volume spanned)
(by $(\bar{A}, \bar{B}, \bar{C})$)

Thus $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}\}$ have

- opposite orientation

4. Boundary Elements and their Orientation.

Example,

Given: An element of volume spanned by the positively oriented triad of vectors (A, B, C):

$$\text{vol} = A \cdot B \times C = \epsilon_{ijk} A^i B^j C^k > 0$$

$$= \epsilon_{ijk} dx^i dx^j dx^k (A, B, C)$$

3!

Question:

- (i a) What are the surface elements which bound this volume?
- (i b) How does one represent them mathematically?
- (ii) Where do they come from?

Answer:

There are 3.2 different boundary elements.
They all come from

$$\text{vol} = \epsilon_{ijk} A^i B^j C^k$$

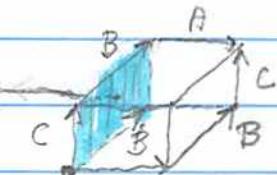
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as follows

$$\text{vol} = A^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k}}_{\text{LEFT}} B^j C^k = A \cdot (\vec{e}_i \epsilon^i_{j k} B^j C^k)$$

LEFT

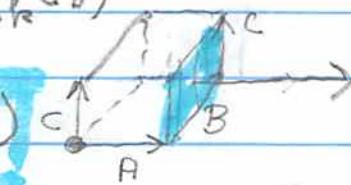
$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (B, C)}$$



$$= (-) A^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k} C^j B^k}_{(-A)} = (-A) \cdot (\vec{e}_i \epsilon^i_{j k} C^j B^k)$$

RIGHT

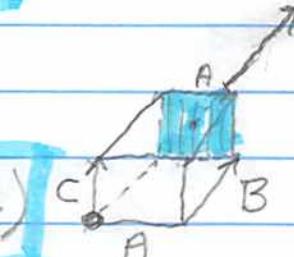
$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (C, B)}$$



$$= B^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k} C^j A^k}_{\text{FRONT}}$$

FRONT

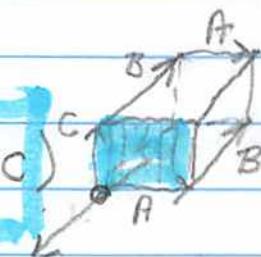
$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (C, A)}$$



$$= (-) B^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k} A^j C^k}_{\text{BACK}}$$

BACK

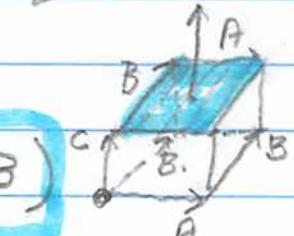
$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (A, C)}$$



$$= C^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k} A^j B^k}_{\text{TOP}}$$

TOP

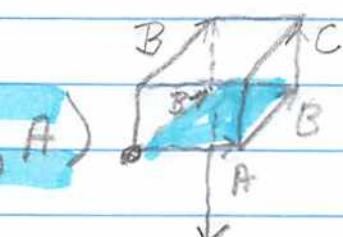
$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (A, B)}$$



$$= (-) C^l \vec{e}_l \cdot \underbrace{\vec{e}_i \epsilon^i_{j k} B^j A^k}_{\text{BOTTOM}}$$

BOTTOM

$$\boxed{\vec{e}_i \epsilon^i_{j k} \frac{dx^j dx^k}{2!} (B, A)}$$

Location
of the face

Conclusion:

The king pin for constructing all $3 \cdot 2 = 6$ boundary elements of

$$\text{vol} = \epsilon_{ijk} \frac{dx^i \wedge dx^j \wedge dx^k}{3!} (A, B, C) = \epsilon_{ijk} A^i B^j C^k$$

is the vectorial 2-form for 3-d space.

$$\Sigma = \vec{e}_i \sum_m^i = \vec{e}_i \cdot d^2 \sum_m^i = \vec{e}_i \cdot \epsilon_{jkl}^i \frac{dx^j \wedge dx^k}{2} \quad \in \binom{1}{2}$$

different notations

It is a multilinear map

$$V \times V \xrightarrow{\Sigma} V$$

$$(A, B) \rightsquigarrow \Sigma(A, B) = \vec{e}_i \cdot \epsilon_{jkl}^i \frac{dx^j \wedge dx^k}{2} (A, B)$$

COMMENT: By applying the above line of reasoning to 4-d spacetime one obtains from the 4-d volume

$$\epsilon_{\alpha\beta\gamma\delta} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta}{4!} (T, X, Y, Z) = \epsilon_{\alpha\beta\gamma\delta} T^\alpha X^\beta Y^\gamma Z^\delta$$

its 8 3-d boundary elements by evaluating

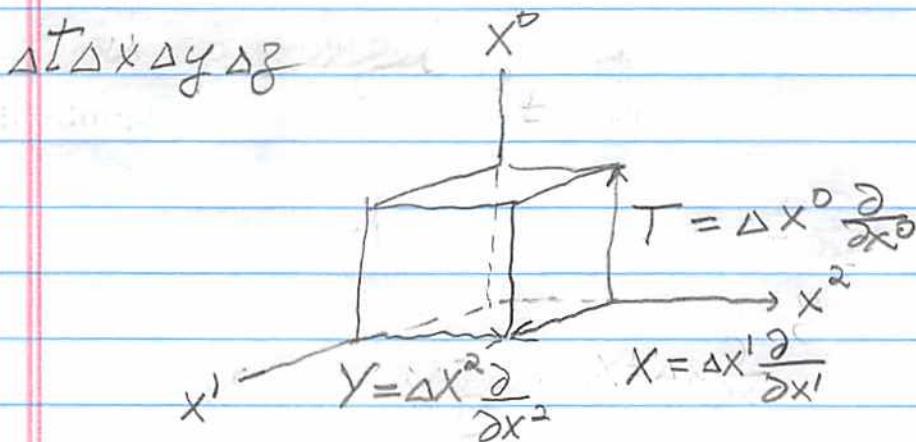
$$\Sigma = e_\sigma d^3 \Sigma \equiv e_\sigma \epsilon^\sigma_{\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!}$$

on the various triads of vectors among (T, X, Y, Z) .

II Momentum Conservation Mathematized by the 3-4 version of Stokes' theorem

Conservation

A volume element $\Delta x \Delta y \Delta z$ during
a time Δt sweeps out a 4-d world
tube whose 4-d spacetime volume



is spanned by the vectors

$$T = T^\alpha \frac{\partial}{\partial x^\alpha}$$

$$X = X^\alpha \frac{\partial}{\partial x^\alpha}$$

$$Y = Y^\alpha \frac{\partial}{\partial x^\alpha}$$

$$Z = Z^\alpha \frac{\partial}{\partial x^\alpha}$$

The 4-d spacetime volume is

$$\sqrt{-g} \Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3 = \epsilon_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta (T, x, y, z)$$

$$= \det \begin{vmatrix} dx^0(T) & dx^1(T) & dx^2(T) & dx^3(T) \\ dx^0(x) & dx^1(x) & dx^2(x) & dx^3(x) \\ dx^0(y) & dx^1(y) & dx^2(y) & dx^3(y) \\ dx^0(z) & dx^1(z) & dx^2(z) & dx^3(z) \end{vmatrix}$$

$$= \epsilon_{\alpha\beta\gamma} T^\alpha x^\alpha y^\beta z^\gamma$$

The amount of momenergy created

in $\Delta x^1 \Delta x^2 \Delta x^3$ during the time interval

$[x^0, x^0 + \Delta x^0]$ is quantified by the creation index Q^0 :

$Q = \text{change in m.e. in } \Delta x^1 \Delta x^2 \Delta x^3 \text{ plus}$

the amount of m.e. flowing out of this

3-d volume through its six sides,

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(its 3 pairs of opposing faces) during time Δx^0 .

Explicitly, in terms of the momenergy tensor components $T^{μν}$, one has

$$Q = e_\mu T^{\mu 0} \sqrt{-g} \Delta x^1 \Delta x^2 \Delta x^3 \Big|_{x^0 + \Delta x^0} - e_\mu T^{\mu 0} \sqrt{-g} \Delta x^1 \Delta x^2 \Delta x^3 \Big|_{x^0}$$

$$+ e_\mu T^{\mu 1} \sqrt{-g} \Delta x^2 \Delta x^3 \Delta x^0 \Big|_{x^0 + \Delta x^1} - e_\mu T^{\mu 1} \sqrt{-g} \Delta x^2 \Delta x^3 \Delta x^0 \Big|_{x^1}$$

+ two more (123)-cycled expressions.

- corresponding to the other two pairs of faces.

$$Q = \nabla_0 \left(e_\mu T^{\mu 0} \sqrt{-g} \right) \Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3$$

$$+ \nabla_1 \left(e_\mu T^{\mu 1} \sqrt{-g} \right) \Delta x^1 \Delta x^2 \Delta x^3 \Delta x^0$$

+ 2 cyclic (123) terms

$$= \nabla_0 \left(e_\mu T^{\mu 0} \sqrt{-g} \right) \Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3$$

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This expression is the exterior derivative

$$d(*T) = d \left(e_{\mu} T^{\mu\rho} e_{\rho\alpha\beta\gamma} \underbrace{dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}}_{3!} \right)$$

evaluated on (T, X, Y, Z) . This is because

$$d*T(T, X, Y, Z) =$$

$$= d(e_{\mu} T^{\mu\rho} e_{\rho\alpha\beta\gamma}) \wedge \underbrace{dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}}_{3!}(T, X, Y, Z)$$

$$= \nabla_{\sigma} (e_{\mu} T^{\mu\rho} e_{\rho\alpha\beta\gamma}) dx^{\sigma} \wedge \underbrace{dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}}_{3!}(T, X, Y, Z)$$

$$= \nabla_{\sigma} (e_{\mu} T^{\mu\rho} e_{\rho\alpha\beta\gamma}) \underbrace{[\delta^{\sigma}_{\alpha} \delta^{\beta}_{\gamma}]}_{3!} dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}(T, X, Y, Z)$$

$$= \nabla_{\sigma} (e_{\mu} T^{\mu\rho} \overline{g^{\sigma\rho}}) \Delta x^{\alpha} \Delta x^{\beta} \Delta x^{\gamma}$$

The Conclusion:

Q = momenergy creation in 4-d volume $e_{\rho\alpha\beta\gamma} T^{\rho} X^{\alpha} Y^{\beta} Z^{\gamma}$

$$= \nabla_{\sigma} (e_{\mu} T^{\mu\rho} \overline{g^{\sigma\rho}}) \Delta x^{\alpha} \Delta x^{\beta} \Delta x^{\gamma}$$

$$= d*T = d(e_{\mu} T^{\mu\rho} e_{\rho\alpha\beta\gamma} \underbrace{dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}}_{3!})(T, X, Y, Z)$$

$$\boxed{Q=0 \Leftrightarrow d*T=0}$$

conservation of momenergy