

LECTURE 15

I) Maxwell's vs Einstein's equations compared and contrasted.

II) The 2-3 version of Stokes' theorem

a) The vectorial version

b) The scalar version

c) The tensorial version.

Reading assignment;

Read these "Lecture 15" notes

and let me know which pages

are unclear and/or not easily

understood.

STOKES' THEOREM

15-1

I) Maxwell's and Einstein's Equations: A Comparison and Contrast.

The advantage of exhibiting Maxwell's equations in integral form is that it gives a more direct connection to what one is mathematizing.

The mathematical method for that integral form are the 1-2 and the 2-3 (Gauss's) versions of Stokes' theorem.

Einstein's equations in their integral formulation have an analogous structure in their integral formulation.

A comparison and contrast between Maxwell and Einstein, we shall see, yields the following observations

(i) Whereas Maxwell's equations mathematize charge conservation,

$$d * J = 0,$$

those of Einstein mathematize momentum energy conservation

$$d * T = 0.$$

(ii) The integral formulation of Einstein's gravitation uses the 2-3 and 3-4

version's of Stokes' theorem instead of Maxwell's which uses the 1-2 and 2-3 version.

(iii) Its mathematical formulation is a reflection of the frame (i.e. coordinate) invariance of the observed phenomena.

II) The 2-3 version of Stokes' theorem 15-3

A) Start by setting up and integrating the surface integral of the vectorial 2-form

$$\vec{\omega}_m = \vec{f} \, dg \wedge dh$$

over the boundary of a 3-d cube in terms of the volume integral over the cube's interior.

B.) Apply this result to the vectorial 2-form

$$\vec{\omega}_m = \vec{e}_i T^{ij} \epsilon_{jkl} \frac{dx^k \wedge dx^l}{2!}$$

which is defined over the bounding surface of that 3-cube, and to its exterior derivative, the 3-form $d\vec{\omega}_m$, which is defined over the volume of that 3-cube.

c) Generalize that method and apply it to the vectorial ("momentum")

3-form

$$*\Pi = e_\sigma T^{\sigma\nu} \epsilon_{\nu\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

That generalization relates

the 4-d spacetime integral of $d*\Pi$ over the $\underbrace{\text{interior}}_{\mathcal{D}}$ of a 4-cube to

the 3-d integrals of $*T$ over the $\underbrace{\text{boundary}}_{\partial\mathcal{D}}$ 8 3-cubes that form the boundary

$\partial\mathcal{D}$ of that 4-d spacetime cube:

$$\int_{\mathcal{D}} d*\Pi = \int_{\partial\mathcal{D}} *T \equiv Q$$

Momentum conservation, $Q=0$, is observed in

any 4-d spacetime domain \mathcal{D} . Consequently,

$$d*\Pi = 0 \quad (\text{differential formulation})$$

$$\int_{\partial\mathcal{D}} *T = 0 \quad (\text{integral formulation})$$

II) The line of reasoning leading to 15.15-5
the 2-3 version of Stokes' theorem

1.) Consider 3-d space coordinatized

by $x^i(t, u, v)$, $i = 1, 2, 3$.

$$\text{Let } \vec{t} = \frac{\partial x^i}{\partial t} \frac{\partial}{\partial x^i} = \frac{\partial}{\partial t}$$

$$\vec{u} = \frac{\partial x^j}{\partial u} \frac{\partial}{\partial x^j} = \frac{\partial}{\partial u}$$

$$\vec{v} = \frac{\partial x^k}{\partial v} \frac{\partial}{\partial x^k} = \frac{\partial}{\partial v}$$

Then

a) the following "infinitesimal" version
of the 2-3 Stokes' theorem is an algebraic
identity which holds at each point $\{x^i(t, u, v)\}$
of the manifold

$$\begin{aligned} \text{(15.1)} \quad \left. \begin{aligned} \text{l.h.s.}(\vec{t}, \vec{u}, \vec{v}) &\equiv \nabla_{\vec{t}} \vec{\Omega}(\vec{u}, \vec{v}) + \nabla_{\vec{u}} \vec{\Omega}(\vec{v}, \vec{t}) + \nabla_{\vec{v}} \vec{\Omega}(\vec{t}, \vec{u}) \\ &- \vec{\Omega}([\vec{u}, \vec{v}], \vec{t}) - \vec{\Omega}([\vec{v}, \vec{t}], \vec{u}) - \vec{\Omega}([\vec{t}, \vec{u}], \vec{v}) \\ &= d \vec{\Omega}(\vec{t}, \vec{u}, \vec{v}) \equiv \text{r.h.s.}(\vec{t}, \vec{u}, \vec{v}) \end{aligned} \right\} \end{aligned}$$

Here ∇ is the covariant derivative
of the vector $\vec{\Omega}(\ , \)$ and $d \vec{\Omega}$ is the exterior
derivative of $\vec{\Omega}$.

b) l.h.s. (, ,) and r.h.s. (, ,) are pointwise
linear in each of their arguments.

Problem for Homework 3

SHOW that Eq.(15.1) holds for all triad of vectors (t, u, v) and hence

SHOW that the l.h.s. is point wise linear in each of its arguments and so is the r.h.s.

2.) Evaluate the r.h.s. of Eq. (15.1):

$$\text{r.h.s.}(t, u, v) = d\vec{\Sigma}(\vec{t}, \vec{u}, \vec{v}) = d\vec{f} \wedge dg \wedge dh(\vec{t}, \vec{u}, \vec{v})$$

Recall that

$$dh(\vec{v}) \equiv \vec{v}(h) \equiv \nabla_{\vec{v}} h \quad \left(\begin{array}{l} \text{"directional"} \\ \text{derivative} \end{array} \right)$$

$$dg(\vec{u}) \equiv \vec{u}(g) \equiv \nabla_{\vec{u}} g \quad \parallel$$

$$d\vec{f}(\vec{t}) \equiv \vec{t}(\vec{f}) \equiv \nabla_{\vec{t}} \vec{f} \quad \left(\begin{array}{l} \text{"cov. directional"} \\ \text{derivative} \end{array} \right)$$

Consequently

$$\text{r.h.s.}(t, u, v) = \nabla_t \vec{f} \nabla_u g \nabla_v h + \text{other even terms}$$

$$- \nabla_u \vec{f} \nabla_t g \nabla_v h + \text{other odd terms}$$

$$= \begin{vmatrix} \nabla_t \vec{f} & \nabla_u \vec{f} & \nabla_v \vec{f} \\ \nabla_t g & \nabla_u g & \nabla_v g \\ \nabla_t h & \nabla_u h & \nabla_v h \end{vmatrix} .$$

3) (i) Specialize to the case where

$$[\vec{E}, \vec{u}] = [\vec{u}, \vec{v}] = [\vec{v}, \vec{E}] = 0,$$

(ii) multiply Eq. (15.1) by $\Delta t \Delta u \Delta v$

and obtain

$$(iii) \mathcal{L}_h \Delta = \nabla_z \vec{\Sigma}(\vec{u}, \vec{v}) \Delta t \Delta u \Delta v$$

$$+ \nabla_u \vec{\Sigma}(\vec{v}, \vec{E}) \Delta t \Delta u \Delta v$$

$$+ \nabla_v \vec{\Sigma}(\vec{E}, \vec{u}) \Delta t \Delta u \Delta v$$

$$= \underbrace{\vec{\Sigma}(\vec{u}, \vec{v}) \Delta u \Delta v}_{(1a)} \Big|_{t+\Delta t} - \underbrace{\vec{\Sigma}(\vec{u}, \vec{v}) \Delta u \Delta v}_{(1b)} \Big|_t$$

where

$$(1a) \alpha \vec{e}_i \cdot \vec{e}_{j'k}^i \frac{dx^{j'} dx^{k'}}{2!} (\vec{u}, \vec{v}) \Delta u \Delta v$$

but

$$(1b) \alpha \vec{e}_i \cdot \vec{e}_{j'k}^i \frac{dx^{j'} dx^{k'}}{2!} (\vec{v}, \vec{u}) \Delta u \Delta v$$



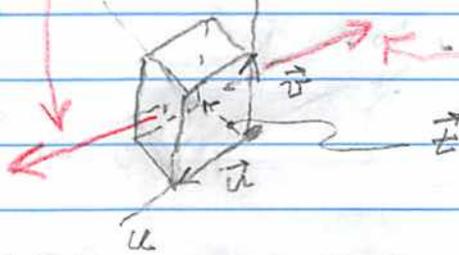
$$+ \underbrace{\vec{\Sigma}(\vec{v}, \vec{E}) \Delta t \Delta v}_{(2a)} \Big|_{u+\Delta u} - \underbrace{\vec{\Sigma}(\vec{v}, \vec{E}) \Delta t \Delta v}_{(2b)} \Big|_u$$

where

$$(2a) \alpha \vec{e}_i \cdot \vec{e}_{j'k}^i \frac{dx^{j'} dx^{k'}}{2!} (\vec{v}, \vec{E}) \Delta t \Delta v$$

but

$$(2b) \alpha \vec{e}_i \cdot \vec{e}_{j'k}^i \frac{dx^{j'} dx^{k'}}{2!} (\vec{E}, \vec{v}) \Delta t \Delta v$$



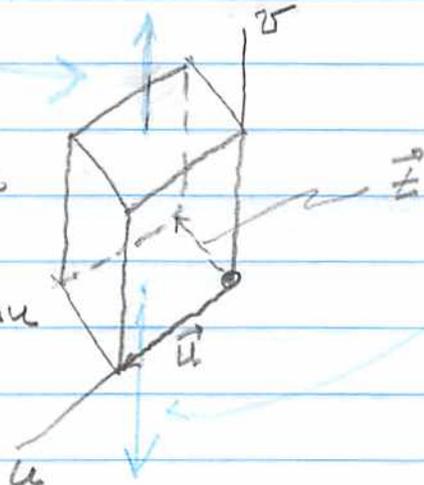
$$+ \underbrace{\vec{\Sigma}(\vec{t}, \vec{u}) \Delta t \Delta u}_{(3a)} \Big|_{v+\Delta v} - \underbrace{\vec{\Sigma}(\vec{t}, \vec{u}) \Delta t \Delta u}_{(3b)} \Big|_v$$

where

$$(3a) \propto \vec{e}_i \epsilon^{ijk} \frac{dx^j dx^k}{2!} (\vec{t}, \vec{u}) \Delta t \Delta u$$

but

$$(3b) \propto \vec{e}_i \epsilon^{ijk} \frac{dx^j dx^k}{2} (\vec{u}, \vec{t}) \Delta t \Delta u$$



$$= d\vec{\Sigma}(\vec{t}, \vec{u}, \vec{v}) \Delta t \Delta u \Delta v$$

4) Summarizing the explicit calculation in 2.) and 3.), obtain

$$(15.2a) \quad \underbrace{(1a) + (1b) + (2a) + (2b) + (3a) + (3b)}_{III} = \underbrace{d\vec{\Sigma}(\vec{t}, \vec{u}, \vec{v}) \Delta t \Delta u \Delta v}_{III}$$

$$(15.2b) \quad \iint_{\partial \mathcal{D}} \vec{\Sigma} = \iiint_{\mathcal{D}} d\vec{\Sigma}$$

Here

$$\mathcal{D} = [t, t + \Delta t] \times [u, u + \Delta u] \times [v, v + \Delta v]$$

is the interior domain of the cube spanned by $(\Delta t \frac{\partial}{\partial t}, \Delta u \frac{\partial}{\partial u}, \Delta v \frac{\partial}{\partial v})$.

5.)

By using the Stokes' theorem one establishes a new unity in the variety of the physical world and in one's mathematization of it.

The two unifying features implicit in this theorem are:

(i) linearity and (ii) that the covariant derivative applies to scalars, vectors, and tensors.

a) Scalars

Being linear, the operations on the l.h.s. and the r.h.s. when applied to the scalar 2-form

$$\Omega = f dg \wedge dh,$$

can be applied to the scalar 2-form

$$\underline{\Omega} = \sum_i \sum_j \sum_k J^i \epsilon_{ijk} dx^j \wedge dx^k$$

without violating the line of reasoning

which starts with Eq. (15.1) and leads

to the conclusion, Eq. (15.2) on page 15-9,

namely

$$\iint_{\partial \mathcal{D}} J^j \epsilon_{jke} \frac{dx^k \wedge dx^e}{2!} = \iiint_{\mathcal{D}} d \left(J^j \epsilon_{jke} \frac{dx^k \wedge dx^e}{2!} \right).$$

A straight forward calculation

(which is part of Problem 3 of Homework 2)

results in

$$d \left(J^j \epsilon_{jke} \frac{dx^k \wedge dx^e}{2} \right) = J^j_{;j} \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3$$

Thus obtain

$$\boxed{\iint_{\partial \mathcal{D}} J^j \epsilon_{jke} \frac{dx^k \wedge dx^e}{2} = \iiint_{\mathcal{D}} J^j_{;j} \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3}$$

which is the corresponding Gauss's theorem for a scalar 2-form.

b) Vectors

15-12

Apply Stokes' theorem as developed on pages 15-7 through 15-9 to the vectorial 2-form

$$\vec{\Omega}_m = \vec{e}_i T^{ij} E_{jke} \frac{dx^k \wedge dx^e}{2!}$$

Using the algebraic line of reasoning on page 14-12 and the result of Problem (2iii) of Homework 2, find that the exterior derivative of $\vec{\Omega}_m$ is

$$(15.3) \begin{cases} d\vec{\Omega}_m = \nabla_i (\vec{e}_i T^{ij} \sqrt{g}) dx^1 \wedge dx^2 \wedge dx^3 & \text{(from page 14-12)} \\ & \text{Lecture 14} \\ d\vec{\Omega}_m = \vec{e}_i T^{ij}{}_{;j} \sqrt{g} dx^1 \wedge dx^2 \wedge dx^3 & \text{(Homework 2)} \end{cases}$$

Insert these into Eq. (15.2),

$$\int_{\partial \mathcal{D}} \underbrace{\vec{e}_i T^{ij} E_{jke}}_{\vec{\Omega}_m} \frac{dx^k \wedge dx^e}{2!} = \int_{\mathcal{D}} d \left(\underbrace{\vec{e}_i T^{ij} E_{jke}}_{d\vec{\Omega}_m} \frac{dx^k \wedge dx^e}{2!} \right)$$

the 2-3 vectorial version of Stokes' theorem,

and, in light of Eq. (15.3), obtain the mathematically equivalent identity

$$\iint_{\partial \mathcal{D}} \vec{e}_i T^{ij} \epsilon_{jke} \frac{dx^k dx^e}{2} = \iiint_{\mathcal{D}} \vec{e}_i T^{ij} \sqrt{g} dx^1 dx^2 dx^3$$

$$= \left(\vec{e}_i T^{ij} \sqrt{g} \Delta t \Delta u \Delta v \right)$$

This is the 2-3 vectorial version of

Gauß's theorem.

Question: what is the scalar-

version?

Answer: For

$$T = T^{\mu\nu} dx^\mu dx^\nu$$

the scalar version of the identity is

$$\iint_{\partial \mathcal{D}} T^{\mu\nu} n_\nu dx^1 dx^2 dx^3 = \iiint_{\mathcal{D}} \partial_\mu T^{\mu\nu} dx^1 dx^2 dx^3$$

the 2-3 scalar

c) Tensors

Instead of the "simple" tensor valued

$$2\text{-form} \quad \leftrightarrow \quad \overleftrightarrow{\Omega} = f dg \wedge dh \in \binom{2}{2}, \quad \mathbb{R}^3$$

consider the "non-simple" tensor
valued 2-form,

$$\overleftrightarrow{\Omega} = \vec{e}_i \otimes \vec{e}_j \cdot R^{ij}_{kl} \frac{dx^k \wedge dx^l}{2!}$$

The line of reasoning page 15-7 through
15-9, which includes reference to
covariant directional derivatives

such as

$$d \overleftrightarrow{f}(\vec{E}) \equiv \nabla_{\vec{E}} \overleftrightarrow{f}$$

leads to

$$\iint_{\partial \mathcal{D}} \vec{e}_i \otimes \vec{e}_j \cdot R^{ij}_{kl} \frac{dx^k \wedge dx^l}{2!} \quad \iint_{\mathcal{D}} d(\vec{e}_i \otimes \vec{e}_j \cdot R^{ij}_{kl} \frac{dx^k \wedge dx^l}{2!})$$