

LECTURE 16

T) Matter in motion: Kinematics

I) Volume as a geometrical attribute

T) Momentum conservation

II) Matter in motion

MTW: Ex. 22.1

III) Change in comoving volume

MTW: §22.3 Matter in motion: Dynamics

MTWi Box 5.5, IV) Monenergy for a perfect fluid

MTW: Ex. 3.18 V) Monenergy for E&M,

MTW: page 155 VI) Conservation of monenergy

A) E&M ("Laser") driven fluid dynamics

I.) VOLUME AS A GEOMETRICAL ATTRIBUTE OF MATTER.

16-1

How does one understand the process of change in matter that moves?

A fundamental aspect of matter is the volume it occupies.

"Volume" is a geometrical attribute of a 3-d domain spanned by spatial displacements into three different directions, which (B.T.W) are directly accessible to one's perceptual faculties ("evidence of the senses").

In 3-d Euclidean space the volume of a domain spanned by an as-yet-unspecified triad of displacements

is mathematized by Levi-Civita tensor

$$\overleftrightarrow{\epsilon} = \epsilon_{ijk} \frac{dx^i \wedge dx^j \wedge dx^k}{3!} = \sqrt{g} dx^i \wedge dx^j \wedge dx^k \in \mathbb{O}(3).$$

For a given triad of vectors, say

$$\vec{w} = \vec{e}_l \Delta w^l, \vec{y} = \vec{e}_m \Delta y^m, \text{ and } \vec{z} = \vec{e}_n \Delta z^n$$

the spanned volume is

$$v = \overleftrightarrow{\epsilon}(\vec{w}, \vec{y}, \vec{z}) = \sqrt{g} [ijk] \Delta w^i \Delta y^j \Delta z^k$$

$$(16.1) \quad = \sqrt{g} \begin{vmatrix} \Delta w^1 & \Delta w^2 & \Delta w^3 \\ \Delta y^1 & \Delta y^2 & \Delta y^3 \\ \Delta z^1 & \Delta z^2 & \Delta z^3 \end{vmatrix} \left(-\vec{x} \cdot \vec{y} \times \vec{z} \right).$$

II.) MATTER IN MOTION

For matter in motion this is the volume

occupying the domain spanned by these

displacements as observed in a comoving frame.

Question: How does one mathematize this

observed volume in a frame of reference relative to which the observed matter has 4-velocity

$$u_i = u^{\mu} e_{\mu} = \frac{\partial x^{\mu}(\tau, \xi^1, \xi^2, \xi^3)}{\partial \tau} \frac{\partial}{\partial x^{\mu}} = \frac{d}{d\tau}$$

$$(\tau, \xi^1, \xi^2, \xi^3)$$

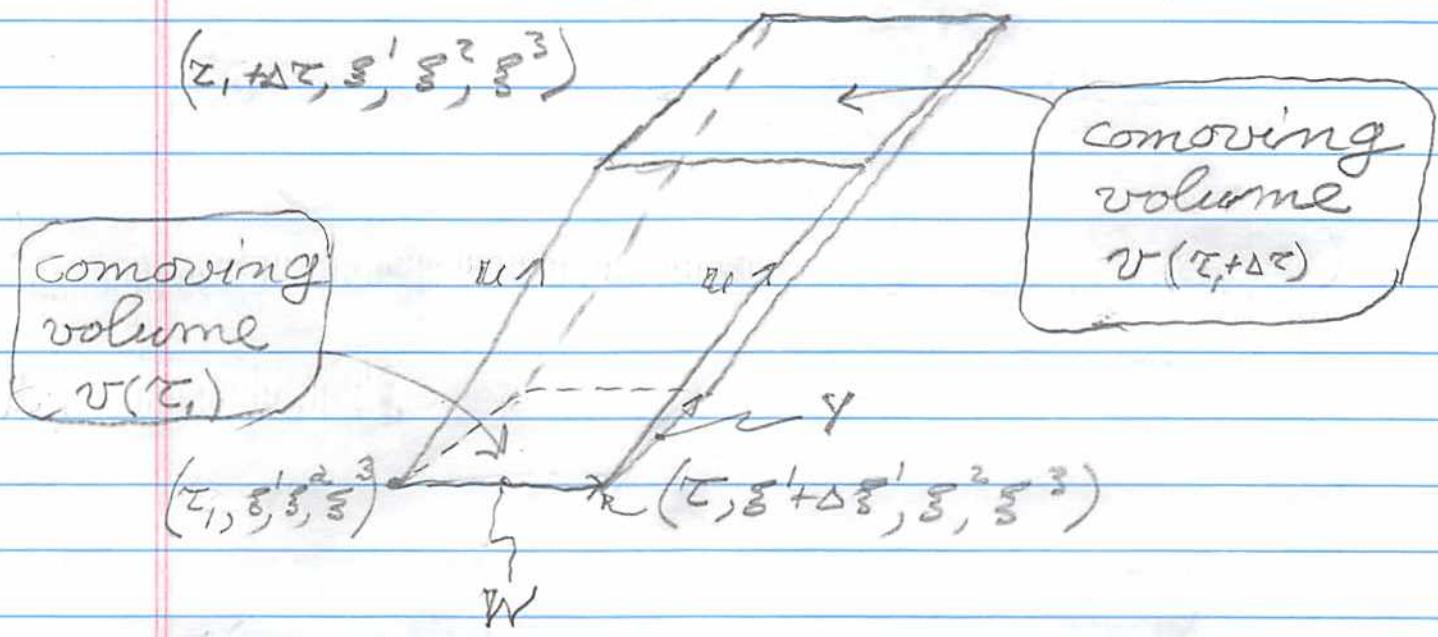


Figure 16.1: The 4-d world cube whose edges are formed from the worldlines of eight particles and whose 4-volume is

$$\Delta \tau \Delta V.$$

Here (ξ^1, ξ^2, ξ^3) are comoving coordinates, which are permanently attached to the particles that comprise the moving matter.

If τ is the proper time of that moving volume of fluid, then the world lines of these comprising particles are coordinatized by

$$\rho(\tau, \xi^i) = \{x^\mu(\tau, \xi^1, \xi^2, \xi^3) : \mu = 0, 1, 2, 3\},$$

The comoving ("proper") 3-volume, Eq. (16.1) on page 16-2, is

$$\begin{aligned} v(\tau) &= u^\mu \epsilon_{\mu\nu\lambda\beta} \frac{dx^\alpha dx^\beta dx^\gamma}{3!} (w, y, z) \\ &= \star u(w, y, z). \end{aligned}$$

III) CHANGE IN COMOVING VOLUME.

Q: What is the change in comoving

3-volume, during $[\tau_i, \tau_i + \Delta\tau]$,

$$v(\tau_i + \Delta\tau) - v(\tau_i) = *u(w, y, z) \Big|_{\tau_i + \Delta\tau} - *u(w, y, z) \Big|_{\tau_i} ?$$

A: Use the 3-4 version of Stokes' theorem,

$$\iiint_{\partial D} *u = \iint_S d(*u),$$

and obtain

$$v(\tau_i + \Delta\tau) - v(\tau_i) = *u(w, y, z) \Big|_{\tau_i + \Delta\tau} - *u(w, y, z) \Big|_{\tau_i}$$

$$\text{zero! } \left\{ \begin{array}{l} +*u(\Delta\tau u, y, z) \Big|_{\text{right}} - *u(\Delta\tau u, y, z) \Big|_{\text{left}} \\ +*u(w, \Delta\tau u, z) \Big|_{\text{front}} - *u(w, \Delta\tau u, z) \Big|_{\text{back}} \\ +*u(w, y, \Delta\tau u) \Big|_{\text{top}} - *u(w, y, \Delta\tau u) \Big|_{\text{bottom}} \end{array} \right.$$

Stokes

$$= d(*u)(\Delta\tau u, w, y, z)$$

$$= u^{\mu}_{;\mu} \underbrace{\sqrt{-g} dx^0 dx^1 dx^2 dx^3}_{\text{comoving volume}} (\Delta x, w, y, z)$$

$$= u^{\mu}_{;\mu} v$$

Conclusion:

$$(16.2) \boxed{\frac{1}{v} \frac{dv}{d\tau} = u^{\mu}_{;\mu}} \quad (= \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} u^{\mu})}{\partial x^{\mu}})$$

i.e the 4-d divergence of the fluid.

4-velocity equals the fractional rate of change in the comoving element of fluid volume.

16-7

IV MOMENERGY for A PERFECT FLUID

Energy density in the comoving frame

$$= \rho(x^\alpha)$$

Isootropic pressure in comoving frame

$$= p(x^\alpha)$$

Four-velocity of each volume element relative to the comoving frame

$$\{u^\mu\} = \{1, 0, 0, 0\}$$

Momenergy tensor components relative the comoving frame

$$[T^{\mu\nu}] = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p & p & p \\ p & p & p & p \end{bmatrix}$$

$$[T_\mu{}^\nu] = g \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + p \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= [p u_\mu u^\nu + p(\delta_\mu^\nu + u_\mu u^\nu)]$$

$$\boxed{T_{\mu}{}^{\nu} = (p+\rho) u_\mu u^\nu + p \delta_\mu^\nu}$$

perfect fluid
relative to
any frame

IV) MOMENERGY for ELECTROMAGNETISM 16-8

The electromagnetic field

$$F = F_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2!}$$

has a momenergy tensor whose components are

$$T_{e.m.}^{\mu\nu} = \frac{1}{4\pi} \left[F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right].$$

The 4-current of a charged fluid with comoving particle density N is a 4-vector whose components are

$$(16.3) \quad J^\mu = qN u^\mu.$$

Here q is the charge per particle so that qN is the comoving charge density.

It is an exercise in "index gymnastics"
(in MTW, Exercise 3.18, page 89)

to show that, in light of the Maxwell field equations, the divergence of the e.m. momenergy tensor

$$(16.4) \quad T_{\text{e.m.},v}^{uv} = -F^{u\alpha} J_\alpha$$

VI CONSERVATION of MOMENERGY.

It is a fact that total momenergy
 $\underbrace{\text{that of}}_{\text{of matter plus electromagnetism}}$
 is neither created nor destroyed.

This is mathematized by the statement

$$(T_{\text{matter}}^{uv} + T_{\text{e.m.}}^{uv})_{;v} = 0$$

or in light of Eqs (16.3) and (16.4) on pages 16-8 and 16-9

$$(16.5) \quad \boxed{T_{\text{matter},v}^{uv} = N g F^{u\alpha} u^\alpha}$$

The right hand side mathematizes

the amount of momenergy injected

into the 4-d spacetime volume $\Delta\tau v$

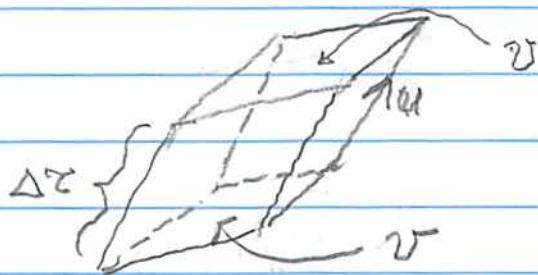


Figure 16.2

The 4-d basis invariant spacetime volume $\Delta\tau v$ is the product of the elapsed proper time $\Delta\tau$ in a comoving elements of fluid whose comoving volume is v .

Indeed, one has

$$Nq F^{\mu}{}_{\alpha} u^{\alpha} = \frac{(\text{particles})}{(\text{comoving volume})} \cdot \frac{(\text{Lorentz})^{\mu}}{(\text{4-force particle})}$$

But recall that

$$\frac{(\text{Lorentz})^{\mu}}{(\text{4-force})} = q F^{\mu}{}_{\alpha} u^{\alpha} = \frac{(\text{momenergy})^{\mu}}{(\text{proper time})} = \frac{dp^{\mu}}{d\tau}$$

Consequently,

$$N \times q F^M_{\alpha} u^{\alpha} = \frac{(\text{monenergy})^M}{(\text{comoving volume})(\text{proper time})}$$

$$= \frac{(\text{monenergy})^M}{(\text{invariant spacetime volume})}$$

Thus Eq. (16,5) mathematizes the amount of mechanical energy created per 4-volume Δv by the electromagnetic field F^M_{α} interacting with the fluid particles each one of which has charge q .

However, a prerequisite for understanding the dynamics of an electromagnetically driven system is that one first grasp its dynamics in the absence of any Maxwell driving forces.