

## LECTURE 17

Relativistic hydrodynamic

A) Generalized Newtonian perspective

MTW: §22.3 B) Energy conservation

MTW: §22.2 C) Chemical potential

MTW: §22.3 D) Euler's 3-d equations

Read MTW §22.2 and §22.3

Comment: For our purposes you may initially set the entropy in these sections equal to zero.

I Being based on the principle of momentum conservation  
 The equations for the relativistic dynamics  
 of a perfect fluid are

$$0 = T^{\sigma\nu}_{;\nu} = \left( g^{\sigma\mu} T_{\mu}^{\nu} \right)_{;\nu} = g^{\sigma\mu}_{;\nu} T_{\mu}^{\nu} + g^{\sigma\mu} T_{\nu}^{\nu}_{;\nu}$$

under zero

or

$$(17.1) \quad 0 = T_{\mu}^{\nu}_{;\nu} = \left[ (p + \rho) u_{\mu} u^{\nu} + p \delta_{\mu}^{\nu} \right]_{;\nu}$$

$$= \underbrace{[(p + \rho) u_{\mu}]_{;\nu} u^{\nu} + (p + \rho) u_{\mu} u^{\nu}_{;\nu}}_{T_{\mu}^{\nu}} + p_{;\nu} \delta_{\mu}^{\nu}$$

$$T_{\mu}^{\nu} = (p + \rho)_{;\nu} u_{\mu} u^{\nu} + (p + \rho) u_{\mu} u^{\nu}_{;\nu}$$

These spacetime equations mathematize four aspects of relativistic fluid dynamics:

- A) Newton's  $F = ma$  perspective, and
- B) Energy conservation,
- C) The concept of a chemical potential when combined with particle conservation.
- D) EULER'S 3-d Equations.

A) Following Newton, consider an element of fluid contained in a comoving volume,  $v$ , whose kinematic evolution follows the law

$$u^{\nu}_{;\nu} = \frac{1}{v} \frac{dv}{dx}.$$

Introduce it into Eq. (17.1)

$$0 = v T_\mu^\nu; \nu = v [(p+g) u_\mu], \nu \frac{dx^\nu}{d\tau} + (p+g) u_\mu \frac{du^\nu}{d\tau} + v p_{,\mu}$$

This equation simplifies to

$$[v(p+g) u_\mu]_{,\nu} \frac{dx^\nu}{d\tau} = -v p_{,\mu}$$

or

$$\nabla_u [v(p+g) u_\mu] = -v \nabla p$$

or in term of the convective derivative

$$(17.2) \quad \frac{D(\ )}{d\tau} = (\ ),_v \frac{dx^\nu}{d\tau},$$

$$\frac{D}{d\tau} [v(p+g) u_\mu] = -v \frac{\partial p}{\partial x^\mu}. \quad \mu = 0, 1, 2, 3$$

Thus the rate of change of generalized fluid 4-momentum in a comoving volume is controlled by the negative 4-d pressure gradient acting present in this volume.

B) Energy conservation is mathematized by

$$u^\mu T_{\mu\nu}^{\;\;\nu} = 0$$

Applying it explicitly to Eq. (17.1) on page 17-1  
one finds that

$$(17.3) \quad u^\mu T_{\mu\nu}^{\;\;\nu} = -(p+g)_{,\nu} u^\nu + \text{zero} - (p+g) u_{;\nu}^{\;\;\nu} + p_{,\mu} u^\mu = 0,$$

where one has used  $u^\mu u_{,\mu} = -1$  and  $u^\mu u_{\mu;\nu} = 0$ .

Again, using the kinematic law, Eq. (16.2) in  
Lecture 16, one finds

$$\text{or} \quad v \frac{dp}{d\tau} + p \frac{dv}{d\tau} = -p \frac{dv}{d\tau}$$

$$(17.4) \quad d(vp) = p(-dv)$$

along every comoving volume element  $v$ .

This equation mathematizes

$$\left( \begin{array}{l} \text{change in energy} \\ \text{in volume} \end{array} \right) = \left( \begin{array}{l} \text{compressional mechanical} \\ \text{work done on the fluid} \\ \text{volume element} \end{array} \right)$$

This is the 1st law of thermodynamics

c) The concept of a chemical potential is mathematized in the context of  $\text{He}$ , particle conservation, i.e., the number of particles in a comoving element of volume is constant,

$$(17.5) \quad \frac{d(Nv)}{dt} = 0 \quad (\text{"particle conservation"})$$

Here  $N = \frac{(\# \text{ of particles})}{(\text{comoving}) \text{ volume}}$

This law is geometrized in terms of the particle 4-velocity by observing that

$$\begin{aligned} 0 &= v \frac{dN}{dt} + N \frac{dv}{dt} \\ &= v \left( N_{;v} u^v + N u^v_{;v} \right) \\ &= v (N u^v)_{;v} \end{aligned}$$

Thus particle conservation, which is

equivalent to

$$\partial = \partial^{\nu} ; v \left( = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{\nu}} (\sqrt{g} \partial^{\nu}) \right),$$

is geometrized by the vanishing of  
the exterior derivative

$$\partial = d(\pm S) = d \left( N u^m e_{\mu \alpha \beta \gamma} dx^{\alpha} dx^{\beta} dx^{\gamma} \right)$$

$$= (N u^m)_{;\mu} \sqrt{-g} dx^0 dx^1 dx^2 dx^3.$$

The fact that the # of particles in volume  $v$   
is constant is restated by

$$v = \frac{\#}{N}$$

Consequently

$$\boxed{\frac{1}{N} = \frac{\text{(mean volume)}}{\text{(particle)}}}$$

Thus the 1<sup>st</sup> law of thermodynamics, Eq.(17.4),  
applied to this context, Eq.(17.5), is

$$\begin{aligned} p &= - \frac{d(v\rho)}{dv} \left( = - \frac{d(\text{energy/particle})}{d(\text{volume/particle})} \right) \\ &= - \frac{d\left(\frac{\rho}{N}\right)}{d\left(\frac{1}{N}\right)} = N \frac{d\rho}{dN} - \sigma \end{aligned}$$

or

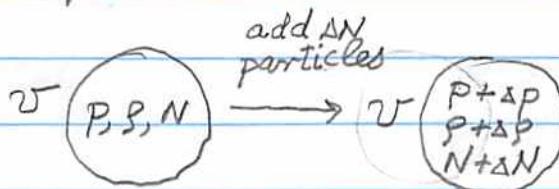
$$\frac{f}{N} + \frac{p}{N} = \frac{\delta f}{\delta N},$$

This is the "chemical potential" of a particle  
*i.e., the injection energy of a particle.*

To arrive at this concept consider a comoving volume element  $v$ , with inside pressure  $p$ , energy density  $f$ , and particle density  $N$ .

Q: How much energy is needed to inject a single particle into this volume?

Simplifying



$$\text{A: } \frac{\Delta f}{\Delta N} = \frac{\frac{\Delta (\text{energy})}{\Delta (\text{unit volume})}}{\frac{\Delta (\# \text{ of particles})}{\Delta (\text{unit volume})}} = \frac{\frac{\Delta (\text{energy})}{v}}{\frac{\Delta (\# \text{ of particles})}{v}} = \frac{\frac{\Delta (\text{energy})}{v}}{\frac{\Delta (\text{injected particles})}{v}} = \frac{\frac{\Delta (\text{injected energy})}{v}}{\Delta (\text{injected particles})}$$

$$= \frac{(\text{energy})}{(\text{particle})} = \frac{(\text{injection energy})}{(\text{particle})} = (" \text{chemical potential} ")$$

Let  $v$  be such that  $\Delta N v = 1$ . In that case  
 $\frac{\Delta f}{\Delta N} = \text{amount of energy necessary to inject}$   
a single particle into the perfect fluid.

Because  $\frac{d\epsilon}{dN} = \frac{P + \delta}{N}$ , this energy consists of two parts

$P \times \frac{1}{N}$  = work necessary to create the volume  $\frac{1}{N}$  to accommodate 1 particle

$\delta \times \frac{1}{N}$  = energy that this particle must have so that, once it occupies the volume, it will be in equilibrium with the surrounding fluid.

D) Obtain the 3-d Euler equations by  
using energy conservation

$$u^\mu T_{\mu\nu}{}^{\nu}_{;\nu} = 0, \text{ Eq. (17.3), to remove } T_{\mu\nu}{}^{\nu}_{;\nu} = 0$$

explicit reference to  $(p+g)u^\nu_{;\nu}$  in  $T_{\mu\nu}{}^{\nu}_{;\nu} = 0$ ,

Eq. (17.1). The resulting equation

$$(p+g)_{,\nu} u_\mu u^\nu + (p+g) u_{\mu;\nu} u^\nu + u_{\mu\nu} p_{,\nu} u^\nu + \delta_{\mu}^{\nu} p_{,\nu} = 0$$

yields

$$(p+g) u_{\mu;\nu} u^\nu = -(\delta_{\mu}^{\nu} + u_\mu u^\nu) p_{,\nu}.$$

These are the relativistic Euler equations. They are purely space like: both the l.h.s. and the r.h.s. have zero components along the 4-velocity  $u = e_\mu u^\mu$ . This is because

$$u^\mu u_\mu = -1 \text{ and } u^\mu u_{\mu;\nu} u^\nu = \frac{1}{2} (u^\mu u_\mu)_{,\nu} = 0$$