or Maxwell'sfield systems would be silent: under such circumstances they do not apply.

$\boxed{\text{1928 Cartan}}$

The problem with Einstein's

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^2}T_{\mu\nu} \tag{1}$$

is that its left hand side is ill-defined when Einstein wrote it down in 1916. He constructed this tensorial equation from the components of the curvature tensor merely to (i) satisfy the momenergy conservation law

$$T^{\mu\nu}{}_{;\mu} = 0$$

and (ii) recover the Newtonian gravitational field equation

$$\nabla^2\phi = 4\pi G\rho \tag{2}$$

in the limit of static weak gravitational fields. Such a construction is necessary but not enough.

In physics and mathematics both sides of an equation (e.g. the stress-strain relation of an elastic madium, F=ma, $\cdots$) must have a well-defined identity. The geometrical meaning of Einstein's l.h.s. and the line of reasoning leading to it need to be specified. Cartan's formulation in 1928, as well Wheeler's in 1964 and in 1990 and Misner's and Wheeler's in 1972, constitute a non-trivial step forward in that direction.

A clue as to the sought-after geometrical meaning of the l.h.s. of Eq.(1) comes from the l.h.s. of Eq.(2). It expresses the following geometrical <u>Proposition:</u>

For a small sphere of

$$\text{volume} = \frac{4\pi r^3}{3}$$

consider the difference between the mean value of ϕ on the surface of this sphere and its value at the center. Then

$$\nabla^2\phi = \text{moment of}\left\{\begin{pmatrix}\text{mean value}\\ \text{on the boundary}\\ \text{of the sphere}\end{pmatrix} - \begin{pmatrix}\text{value at}\\ \text{the center}\\ \text{of the sphere}\end{pmatrix}\right\}\frac{8\pi}{\text{volume}}. \tag{3}$$

(Comment: This property of $\nabla^2\phi$ was pointed out by Maxwell already in 1881.) The validity of Eq.(3) is based on the following mathematical reasoning: Consider the mean value (M.V.) of the difference of the potential ϕ on the surface and the center of a small sphere of surface area $4\pi r^2$,

$$\text{M.V.} = \frac{1}{4\pi r^2}\int_0^{\pi}\int_0^{2\pi}[\phi(x,y,z) - \phi(0,0,0)]r^2 \sin\theta \, d\theta \, d\varphi \tag{4}$$

Expand $[\phi(x,y,z) - \phi(0,0,0)]$ to second order and obtain

$$
\begin{aligned}
[\cdots] =& x\,\phi_{,x} + y\,\phi_{,y} + z\,\phi_{,z} && (5)\\
&+ \frac{1}{2}x^2\phi_{,xx} + \frac{1}{2}y^2\phi_{,yy} + \frac{1}{2}z^2\phi_{,zz} && (6)\\
&+ xy\,\phi_{,xy} + yz\,\phi_{,yz} + zx\,\phi_{,zx}, && (7)
\end{aligned}
$$

where all partial derivatives are evaluated at the center $(0,0,0)$ of the sphere. Introduce the spherical coordinates for (x,y,z), do the integration and finf that all linear and off-diagonal quadratic terms integrate to zero. The diagonal terms involving x^2, y^2, and z^2 yield identical results. One obtains

$$
(\text{M.V.}) = \frac{1}{4\pi r^2}\frac{1}{2}\frac{4\pi r^4}{3}\,\left(\phi_{,xx} + \phi_{,yy} + \phi_{,zz}\right)\big|_{(0,0,0)}\,. \tag{8}
$$

Consequently, the moment of this mean deviation is

$$
r \times (\text{M.V.}) = \frac{1}{8\pi}(\text{volume})\,\nabla^2\phi\Big|_{(0,0,0)} \tag{9}
$$

or

$$
\nabla^2\phi = r \times (\text{M.V.})\frac{8\pi}{(\text{volume})} \tag{10}
$$

Compare this moment of mean value expression with Newton's gravitational field equation Eq.(2). One obtains for a small sphere of radius r

$$
r \times (\text{M.V.} = \frac{G}{2}\rho\,(\text{volume}) = \frac{G}{2}(\text{mass}) \tag{11}
$$

$$
r \times \{16\pi(\text{M.V.}\} = \frac{8\pi G}{c^2}(\text{mass}\cdot c^2) \tag{12}
$$

Thus, Newton's gravitational field equation integrated over a sphere of volume $\frac{4\pi r^3}{3}$ is

$$
\text{moment of}\left\{16\pi\left[\begin{array}{c}\text{deviation of}\\ \text{the surface}\\ \text{mean value of}\\ \text{the gravitational}\\ \text{potential on the}\\ \text{boundary of a}\\ \text{3-volume away}\\ \text{from it6s value}\\ \text{at the center}\\ \text{of that 3-volume}\end{array}\right]\right\} = \frac{8\pi G}{c^2}\left(\begin{array}{c}\text{amount of mass}\cdot \mathrm{c}^2\\ \text{inside that 3-volume}\end{array}\right) \tag{13}
$$

This is the integral formulation of the Newtoniuan differential field Eq.(2). This equation relates what is in the interior of a volume to the moment of something that is measurable on its boundary.

Cartan, Misner, and Wheeler generalize this moment-based feature to 3-cubes[1] in spacetime. With them the integral formulation of the Einstein field equations get geometrized into the form

$$\text{sum of moments of} \left\{ \begin{array}{c} \text{rotation for} \\ \text{the 6 faces} \\ \text{of a small} \\ \text{3-cube} \end{array} \right\} = \frac{8\pi G}{c^2} \left(\begin{array}{c} \text{amount of} \\ \text{momenergy} \\ \text{inside this} \\ \text{3-cube} \end{array} \right) \tag{14}$$

(a) Each 3-cube has associated with it (i) a geometrical object, its <u>total moment of rotation</u> and (ii) a certain vectorial <u>amount of momenergy</u>, which it occupies.

(b) The field Eqs.(1) state that the momenergy in the 3-cube is the source of the moment of rotation. Both are collinjear 4-vectors with Newton's relativized gravitational constant $\frac{8\pi G}{c^2}$ as the constant of proportionality.

(c) Each vector has 4 momenergy components, and there are four 3-cube components ($\Delta x\Delta y\Delta z$, $\Delta t\Delta y\Delta z$, $\Delta t\Delta z\Delta x$, $\Delta t\Delta x\Delta y$) for each volume 4-vector. Consequently, there are $4 \times 4 = 16$ equations as compared to only one equation for the Newtonian gravitation.

(d) To summarize: these equations say that, except for the universal factor $8\pi G/c^2$, the quantity of moment of rotation equals the amount of momenergy in each of these 3-cubes.

[1] more precisely, to the components $(\Delta x\Delta y\Delta z)\,,(\Delta t\Delta y\Delta z)\,,(\Delta t\Delta z\Delta x)\,,(\Delta t\Delta x\Delta y)$ of a typical "volume vector"